54. (a) $f(2) = x^2 - a^2 - x$

$$= (2)^2 - a^2 2$$
$$= 4 - 2a^2$$

(b) $f(2) = 4 - 2x^2$

$$= 4 - 2(2)^2$$
$$= 4 - 8 = -4$$

(c) For $x \neq 2, f$ is continuous. For $x = 2$, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) = -4 \text{ as long as } a = \pm 2.$$

55. (a) $g(x) = \dfrac{x^3}{x^2} = x$

(b) $\dfrac{f(x)}{g(x)} = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3} = \dfrac{1}{x}$

$$= \dfrac{x^3 - 2x^2 + 1}{x^3 + 3x}$$

$$\dfrac{x^3}{x^3} = 1$$

Chapter 3
Derivatives

Section 3.1 Derivative of a Function
(pp. 99–108)

Exploration 1   Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since y is measured in inches and x is measured in days, the derivative $\dfrac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the y-axis in Figure 3.3b.

2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of 5 inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

5. The y-coordinate of point C gives the maximum depth of the water level in the ditch over the 7-day period, while the x-coordinate of C gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at C', indicating that this is when the water level stops rising and begins falling.

6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

Quick Review 3.1

1. $\lim\limits_{h \to 0} \dfrac{(2+h)^2 - 4}{4} = \lim\limits_{h \to 0} \dfrac{(4 + 4h + h^2) - 4}{h}$

$$= \lim_{h \to 0} 4 + h$$
$$= 4 + 0 = 4$$

2. $\lim\limits_{x \to 2^+} \dfrac{x+3}{2} = \dfrac{2+3}{2} = \dfrac{5}{2}$

3. Since $\dfrac{|y|}{y} = -1$ for $y < 0$, $\lim\limits_{y \to 0^-} \dfrac{|y|}{y} = -1$.

4. $\lim\limits_{x \to 4} \dfrac{2x - 8}{\sqrt{x} - 2} = \lim\limits_{x \to 4} \dfrac{2(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2}$

$$= \lim_{h \to 4} 2(\sqrt{x} + 2) = 2(\sqrt{4} + 2) = 8$$

5. The vertex of the parabola is at $(0, 1)$. The slope of the line through $(0, 1)$ and another point $(h, h^2 + 1)$ on the parabola is $\dfrac{(h^2 + 1) - 1}{h - 0} = h$. Since $\lim\limits_{h \to 0} h = 0$, the slope of the line tangent to the parabola at its vertex is 0.

6. Use the graph of f in the window $[-6, 6]$ by $[-4, 4]$ to find that $(0, 2)$ is the coordinate of the high point and $(2, -2)$ is the coordinate of the low point. Therefore, f is increasing on $(-\infty, 0]$ and $[2, \infty)$.

7. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 1)^2 = (1 - 1)^2 = 0$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x + 2) = 1 + 2 = 3$

8. $\lim\limits_{h \to 0^+} f(1 + h) = \lim\limits_{x \to 1^+} f(x) = 0$

9. No, the two one-sided limits are different (see Exercise 7).

10. No, f is discontinuous at $x = 1$ because $\lim\limits_{x \to 1} f(x)$ does not exist.

Section 3.1 Exercises

1. $\dfrac{d}{dx}f(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\dfrac{-h}{x^2+hx}\dfrac{1}{h}$$

$$=\lim_{h\to 0}\dfrac{-h}{x^2h+h^2x}=\lim_{h\to 0}-\dfrac{1}{x^2+hx}$$

$$=-\dfrac{1}{x^2}-\dfrac{1}{(2)^2}=-\dfrac{1}{4}$$

2. $\dfrac{d}{dx}f(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\dfrac{(x+h)^2+4-(x^2+4)}{h}$$

$$=\lim_{h\to 0}\dfrac{2xh+h^2}{h}=\lim_{h\to 0}2x+h=2x$$

$$2(1)=2$$

3. $\dfrac{d}{dx}f(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\dfrac{3-(x+h)^2-(3-x^2)}{h}$$

$$=\lim_{h\to 0}\dfrac{-2xh-h^2}{h}$$

$$=\lim_{h\to 0}-2x-h=-2x$$

$$-2(-1)=2$$

4. $\dfrac{d}{dx}f(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\dfrac{(x+h)^3+(x+h)-(x^3+x)}{h}$$

$$=\lim_{h\to 0}\dfrac{3x^2h+3xh^2+h^3+h}{h}$$

$$=\lim_{h\to 0}3x^2+3xh+h^2+1$$

$$=3x^2+1$$

$$3(0)^2+1=1$$

5. $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$$=\lim_{x\to a}\dfrac{\frac{1}{x}-\frac{1}{a}}{x-a}$$

$$=\lim_{x\to a}\dfrac{a-x}{ax}\left(\dfrac{1}{x-a}\right)$$

$$=\lim_{x\to a}-\dfrac{1}{ax}=-\dfrac{1}{a^2}$$

$$-\dfrac{1}{(2)^2}=-\dfrac{1}{4}$$

6. $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$$=\lim_{x\to a}\dfrac{x^2+4-a^2-4}{x-a}$$

$$=\lim_{x\to a}\dfrac{x^2-a^2}{x-a}$$

$$=\lim_{x\to a}x+a=2a$$

$$2(1)=2$$

7. $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$$=\lim_{x\to a}\dfrac{\sqrt{x+1}-\sqrt{a+1}}{x-a}$$

$$=\lim_{x\to a}\dfrac{x-a}{x-a(\sqrt{x+1}+\sqrt{a+1})}$$

$$=\lim_{x\to a}\dfrac{1}{2\sqrt{a+1}}=\dfrac{1}{2\sqrt{3}+1}$$

$$=\dfrac{1}{4}$$

8. $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$$=\lim_{x-a}\dfrac{2x+3-(2a+3)}{x-a}$$

$$=\lim_{x-a}\dfrac{2x-2a}{x-a}=2$$

9. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\dfrac{[3(x+h)-12]-(3x-12)}{h}$$

$$=\lim_{h\to 0}\dfrac{3h}{h}=\lim_{h\to 0}3=3$$

10. $\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{y(x+h)-y(x)}{h}$

$$=\lim_{h\to 0}\dfrac{7(x+h)-7x}{h}$$

$$=\lim_{h\to 0}\dfrac{7h}{h}=\lim_{h\to 0}7=7$$

11. Let $f(x)=x^2$

$$\dfrac{d}{dx}(x^2)=f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\dfrac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}(2x+h)=2x$$

12. $\dfrac{d}{dx} f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3(x+h)^2 - 3x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + h^2 - 3x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{6xh + h^2}{h} = \lim\limits_{h \to 0} 6x + h = 6x$

13. The graph of $y = f_1(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so its derivative is negative for $x < 0$ and positive for $x > 0$. (b)

14. The graph of $y = f_2(x)$ is always increasing, so its derivative is always ≥ 0. (a)

15. The graph of $y = f_3(x)$ oscillates between increasing and decreasing, so its derivative oscillates between positive and negative. (d)

16. The graph of $y = f_4(x)$ is decreasing, then increasing, then decreasing, and then increasing, so its derivative is negative, then positive, then negative, and then positive. (c)

17. (a) The tangent line has slope 5 and passes through (2, 3).
$y = 5(x - 2) + 3$
$y = 5x - 7$

(b) The normal line has slope $-\dfrac{1}{5}$ and passes through $(2, 3)$.

$y = -\dfrac{1}{5}(x - 2) + 3$

$y = -\dfrac{1}{5}x + \dfrac{17}{5}$

18. $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{y(x+h) - y(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[2(x+h)^2 - 13(x+h) + 5] - (2x^2 - 13x + 5)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$

$= \lim\limits_{h \to 0} \dfrac{4xh + 2h^2 - 13h}{h}$

$= \lim\limits_{h \to 0} (4x + 2h - 13) = 4x - 13$

At $x = 3$, $\dfrac{dy}{dx} = 4(3) - 13 = -1$, so the tangent line has

slope -1 and passes through $(3, y(3)) = (3, -16)$.
$y = -1(x - 3) - 16$
$y = -x - 13$

19. Let $f(x) = x^3$.

$f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(1+h)^3 - 1^3}{h}$

$= \lim\limits_{h \to 0} \dfrac{1 + 3h + 3h^2 + h^3 - 1}{h}$

$= \lim\limits_{h \to 0} (3 + 3h + h^2) = 3$

(a) The tangent line has slope 3 and passes through (1, 1).
Its equation is $y = 3(x - 1) + 1$, or $y = 3x - 2$.

(b) The normal line has slope $-\dfrac{1}{3}$ and passes through (1, 1).

Its equation is $y = -\dfrac{1}{3}(x - 1) + 1$, or $y = -\dfrac{1}{3}x + \dfrac{4}{3}$.

20. Let $f(x) = \sqrt{4}$

$f'(4) = \lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(4+h)^{1/2} - \sqrt{4}}{h}$

$= \lim\limits_{h \to 0} \dfrac{4 + h - 4}{h((4+h)^{1/2} + 2)}$

$= \lim\limits_{h \to 0} \dfrac{h}{h(4+h)^{1/2} + 2} = \dfrac{1}{4}$

(a) The tangent line has slope $\dfrac{1}{4}$ and passes through $(4, 2)$.

Its equation is $y = \dfrac{1}{4}x + 1$.

(b) The normal line has slope -4 and passes through (4, 2).
Its equation is $y = -4x + 18$.

21. (a) The amount of daylight is increasing at the fastest rate when the slope of the graph is largest. This occurs about one-fourth of the way through the year, sometime around April 1. The rate at this time is approximately

$\dfrac{4 \text{ hours}}{24 \text{ days}}$ or $\dfrac{1}{6}$ hour per day.

(b) Yes, the rate of change is zero when the tangent to the graph is horizontal. This occurs near the beginning of the year and halfway through the year, around January 1 and July 1.

(c) Positive: January 1 through July 1
Negative: July 1 through December 31

22. The slope of the given graph is zero at $x \approx -2$ and at $x \approx 1$, so the derivative graph includes $(-2, 0)$ and $(1, 0)$. The slopes at $x = -3$ and at $x = 2$ are about 5 and the slope at $x = -0.5$ is about -2.5, so the derivative graph includes $(-3, 5)$, $(2, 5)$, and $(-0.5, -2.5)$. Connecting the points smoothly, we obtain the graph shown.
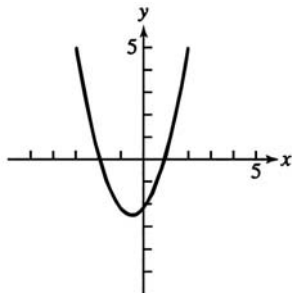
23. (a) Using Figure 3.10a, the number of rabbits is largest after 40 days and smallest from about 130 to 200 days. Using Figure 3.10b, the derivative is 0 at these times.

(b) Using Figure 3.10b, the derivative is largest after 20 days and smallest after about 63 days. Using Figure 3.10a, there were 1700 and about 1300 rabbits, respectively, at these times.

24. Since the graph of $y = x \ln x - x$ is decreasing for $0 < x < 1$ and increasing for $x > 1$, its derivative is negative for $0 < x < 1$ and positive for $x > 1$. The only one of the given functions with this property is $y = \ln x$. Note also that $y = \ln x$ is undefined for $x < 0$, which further agrees with the given graph. (ii)

25. Each of the functions $y = \sin x$, $y = x$, $y = \sqrt{x}$ has the property that $y(0) = 0$ but the graph has nonzero slope (or undefined slope) at $x = 0$, so none of these functions can be its own derivative. The function $y = x^2$ is not its own derivative because $y(1) = 1$ but

$$y'(1) = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0}(2 + h) = 2.$$

This leaves only e^x, which can plausibly be its own derivative because both the function value and the slope increase from very small positive values to very large values as we move from left to right along the graph. (iv)

26. (a) The slope from $x = -4$ to $x = 0$ is $\dfrac{2 - 0}{0 - (-4)} = \dfrac{1}{2}$.
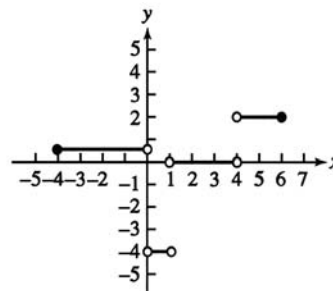
The slope from $x = 0$ to $x = 1$ is $\dfrac{-2 - 2}{1 - 0} = -4$.

The slope from $x = 1$ to $x = 4$ is $\dfrac{-2 - (-2)}{4 - 1} = 0$.

The slope from $x = 4$ to $x = 6$ is $\dfrac{2 - (-2)}{6 - 4} = 2$.

Note that the derivative is undefined at $x = 0$, $x = 1$, and $x = 4$. (The function is differentiable at $x = -4$ and at
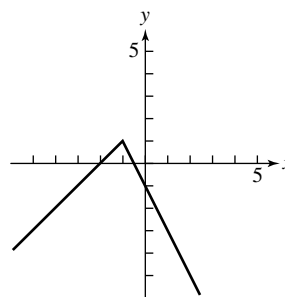
$x = 6$ because these are endpoints of the domain and the one-sided derivatives exist.) The graph of the derivative is shown.

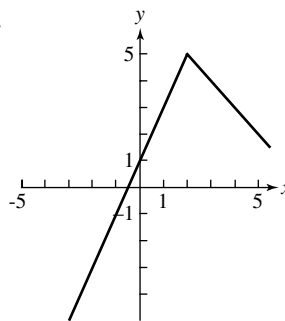**(b)** $x = 0, 1, 4$

27. For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope -2 that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

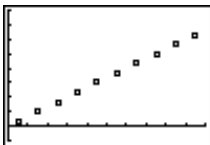Thus $f(x) = \begin{cases} x + 2, & x < -1 \\ -2x - 1, & x \geq -1 \end{cases}$

28.

29.

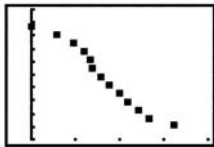| Midpoint of Interval (x)	Slope $\left(\dfrac{\Delta y}{\Delta x}\right)$
0.5	$\dfrac{3.3-0}{1-0}=3.3$
1.5	$\dfrac{13.3-3.3}{2-1}=10.0$
2.5	$\dfrac{29.9-13.3}{3-2}=16.6$
3.5	$\dfrac{53.2-29.9}{4-3}=23.3$
4.5	$\dfrac{83.2-53.2}{5-4}=30.0$
5.5	$\dfrac{119.8-83.2}{6-5}=36.6$
6.5	$\dfrac{163.0-119.8}{7-6}=43.2$
7.5	$\dfrac{212.9-163.0}{8-7}=49.9$
8.5	$\dfrac{269.5-212.9}{9-8}=56.6$
9.5	$\dfrac{332.7-269.5}{10-9}=63.2$

A graph of the derivative data is shown.

[0,10] by [–10,80]

(a) The derivative represents the speed of the skier.

(b) Since the distances are given in feet and the times are given in seconds, the units are feet per second.

(c) The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is

$\dfrac{63.2-0}{9.5-0}\approx 6.65.$ The equation of the derivative is

approximately $D=6.65t$.

30. (a)

[−0.5, 4] by [700, 1700]

(b)

Midpoint of Interval (x)	Slope $\left(\dfrac{\Delta y}{\Delta x}\right)$
$\dfrac{0.00+0.56}{2}=0.28$	$\dfrac{1512-1577}{0.56-0.00}\approx -116.07$
$\dfrac{0.56+0.92}{2}=0.74$	$\dfrac{1448-1512}{0.92-0.56}\approx -177.78$
$\dfrac{0.92+1.19}{2}=1.055$	$\dfrac{1384-1448}{1.19-0.92}\approx -237.04$
$\dfrac{1.19+1.30}{2}=1.245$	$\dfrac{1319-1384}{1.30-1.19}\approx -590.91$
$\dfrac{1.30+1.39}{2}=1.345$	$\dfrac{1255-1319}{1.39-1.30}\approx -711.11$
$\dfrac{1.39+1.57}{2}=1.48$	$\dfrac{1191-1255}{1.57-1.39}\approx -355.56$
$\dfrac{1.57+1.74}{2}=1.655$	$\dfrac{1126-1191}{1.74-1.57}\approx -382.35$
$\dfrac{1.74+1.98}{2}=1.86$	$\dfrac{1062-1126}{1.98-1.74}\approx -266.67$
$\dfrac{1.98+2.18}{2}=2.08$	$\dfrac{998-1062}{2.18-1.98}=-320.00$
$\dfrac{2.18+2.41}{2}=2.295$	$\dfrac{933-998}{2.41-2.18}\approx -282.61$
$\dfrac{2.41+2.64}{2}=2.525$	$\dfrac{869-933}{2.64-2.41}\approx -278.26$
$\dfrac{2.64+3.24}{2}=2.94$	$\dfrac{805-869}{3.24-2.64}\approx -106.67$

A graph of the derivative data is shown.
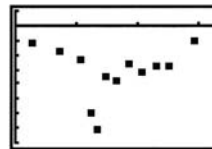
[0, 3.24] by [−800, 100]

(c) Since the elevation y is given in feet and the distance x down river is given in miles, the units of the gradiant are feet per mile.

(d) Since the elevation y is given in feet and the distance x downriver is given in miles, the units of the distance $\dfrac{dy}{dx}$ are feet per mile.

(e) Look for the steepest part of the curve. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

(f) Look for the lowest point on the graph. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

31. $f(x) = \lim\limits_{h\to0^-} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to0^+} \dfrac{3(1+h)-2-2}{h}$

$= \lim\limits_{h\to0^+} \dfrac{3h-1}{h} = -\infty$

Does not exist.

32. We show that the right-hand derivative at 1 does not exist.

$\lim\limits_{h\to0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to0^+} \dfrac{3(1+h)-(1)^3}{h}$

$= \lim\limits_{h\to0^+} \dfrac{2+3h}{h} = \lim\limits_{h\to0^+}\left(\dfrac{2}{h}+3\right) = \infty$

33.
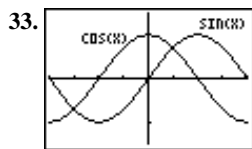
[−π, π] by [−1.5, 1.5]

The cosine function could be the derivative of the sine function. The values of the cosine are positive where the sine is increasing, zero where the sine has horizontal tangents, and negative where sine is decreasing.
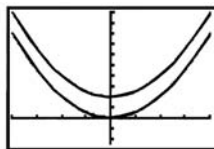
34. $\lim\limits_{h\to0^+} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to0^+} \dfrac{\sqrt{h}-\sqrt{0}}{h} = \lim\limits_{h\to0^+} \dfrac{\sqrt{h}}{h}$

$= \lim\limits_{h\to0^+} \dfrac{1}{\sqrt{h}} = \infty$

Thus, the right-hand derivative at 0 does not exist.

35. Two parabolas are parallel if they have the same derivative at every value of x. This means that their tangent lines are parallel at each value of x.

Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.

[−4, 4] by [−5, 20]

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

36. True. $f'(x) = 2x + 1$

37. False. Let $f(x) = \dfrac{|x|}{x}$. The left hand derivative at $x = 0$ is −1 and the right hand derivative at $x = 0$ is 1. $f'(0)$ does not exist.

38. C. $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to0} \dfrac{4-3(x+h)-(4+3x)}{h}$

$= \lim\limits_{h\to0} \dfrac{-3h}{h} = -3$

39. A. $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to0} \dfrac{1-3(x+h)^2-(1-3x^2)}{h}$

$= \lim\limits_{h\to0} \dfrac{-6xh-3h^2}{h} = \lim\limits_{h\to0} 6x+h = -6x$

$-6(1) = -6.$

40. B. $\lim\limits_{h\to0^-} \dfrac{(0-1+h)^2-(0-1)}{h}$

$= \lim\limits_{h\to0^-} \dfrac{2-2h+h^2}{h} = \lim\limits_{h\to0} h = 0$

41. C. $\lim\limits_{h\to0^+} \dfrac{(2(0+h)-1)-(2(0)-1)}{h}$

$= \lim\limits_{h\to0^+} \dfrac{2h}{h} = 2$

42. (a) $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to0} \dfrac{(x+h)^2-x^2}{h} = \lim\limits_{h\to0} \dfrac{2xh+h^2}{h}$

$= \lim\limits_{h\to0}(2x+h) = 2x$

(b) $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to0} \dfrac{2(x+h)-2x}{h} = \lim\limits_{h\to0} 2 = 2$

(c) $\lim\limits_{x\to1^-} f'(x) = \lim\limits_{x\to1^-} 2x = 2(1) = 2$

(d) $\lim\limits_{x\to1^+} f'(x) = \lim\limits_{x\to1^+} 2 = 2$

(e) Yes, the one-sided limits exist and are the same, so $\lim\limits_{x\to1} f'(x) = 2$.

(f) $\lim\limits_{h\to0^-} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{(1+h)^2-1^2}{h}$

$= \lim\limits_{h\to0^-} \dfrac{2h+h^2}{h} = \lim\limits_{h\to0^-}(2+h) = 2$

(g) $\lim\limits_{h\to0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{2(1+h)-1^2}{h}$

$= \lim\limits_{h\to0^-} \dfrac{1+2h}{h} = \lim\limits_{h\to0^-}\left(\dfrac{1}{h}+2\right) = -\infty$

The right-hand derivative does not exist.

42. Continued

(h) It does not exist because the right-hand derivative does not exist.

43. The y-intercept of the derivative is $b - a$.

44. Since the function must be continuous at $x = 1$, we have
$$\lim_{x \to 1^+} (3x + k) = f(1) = 1, \text{ so } 3 + k = 1, \text{ or } k = -2.$$

This gives $f(x) = \begin{cases} x^3, & x \le 1 \\ 3x - 2, & x > 1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^3 - (1)^3}{h}$$
$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h}$$
$$= \lim_{h \to 0} (3 + 3h + h^2) = 3$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[3(1+h) - 2] - (1)^3]}{h}$$

$$= \lim_{h \to 0^+} \frac{(1 + 3h) - 1}{h} = \lim_{h \to 0^+} 3 = 3$$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

45. (a) $1 \cdot \dfrac{364}{365} \cdot \dfrac{363}{365} \approx 0.992$

Alternate method: $\dfrac{_{365}P_3}{365^3} \approx 0.992$

(b) Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

(c) Let P represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is

$(1 - P)\left(\dfrac{362}{365}\right)$, so the probability of a shared birthday is

$1 - (1 - P)\left(\dfrac{362}{365}\right) \approx 0.016.$

(d) No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

Section 3.2 Differentiability (pp. 109–115)

Exploration 1   Zooming in to "See" Differentiability

1. Zooming in on the graph of f at the point (0, 1) always produces a graph exactly like the one shown below,
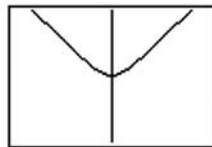
provided that a square window is used. The corner shows no sign of straightening out.
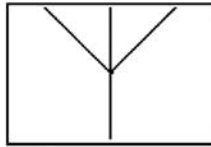
[−0.25, 0.25] by [0.836, 1.164]

2. Zooming in on the graph of g at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window.
[−4, 4] by [−1.624, 3.624].

[−0.0625, 0.0625] by [0.959, 1.041]

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.
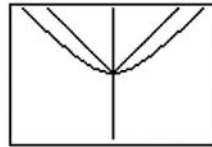
[−0.03125, 0.03125] by [0.9795, 1.0205]

Exploration 2   Looking at the Symmetric Difference Quotient Analytically

1. $\dfrac{f(10 + h) - f(10)}{h} = \dfrac{(10.01)^2 - 10^2}{0.01} = 20.01$
$f'(10) = 2 \cdot 10 = 20$
The difference quotient is 0.01 away from $f'(10)$.

2. $\dfrac{f(10 + h) - f(10 - h)}{2h} = \dfrac{(10.01)^2 - (9.99)^2}{0.02} = 20$
The symmetric difference quotient exactly equals $f'(10)$.

3. $\dfrac{f(10 + h) - f(10)}{h} = \dfrac{(10.01)^3 - 10^3}{0.01} = 300.3001.$
$f'(10) = 3 \cdot 10^2 = 300$
The difference quotient is 0.3001 away from $f'(10)$.

$\dfrac{f(10 + h) - f(10 - h)}{2h} = \dfrac{(10.01)^3 - (9.99)^3}{0.02} = 300.0001.$ The symmetric difference quotient is 0.0001 away from $f'(10)$.

Quick Review 3.2

1. Yes

2. No (The $f(h)$ term in the numerator is incorrect.)

3. Yes   **4.** Yes

5. No (The denominator for this expression should be $2h$).

6. All reals

7. $[0, \infty)$ **8.** $[3, \infty)$

9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.

10. $\dfrac{f(3+0.001) - f(3-0.001)}{0.002} = \dfrac{5(3+0.001) - 5(3-0.001)}{0.002}$

$= \dfrac{5(0.002)}{0.002} = 5$

Section 3.2 Exercises

1. Left-hand derivative:

$\lim\limits_{h\to0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to0^-} \dfrac{h^2 - 0}{h} = \lim\limits_{h\to0^-} h = 0$

Right-hand derivative:

$\lim\limits_{h\to0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to0^+} \dfrac{h - 0}{h} = \lim\limits_{h\to0^+} 1 = 1$

Since $0 \ne 1$, the function is not differentiable at the point P.

2. Left-hand derivative:

$\lim\limits_{h\to0^-} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{2 - 2}{h} = \lim\limits_{h\to0^-} 0 = 0$

Right-hand derivative:

$\lim\limits_{h\to0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^+} \dfrac{2(1+h) - 2}{h} = \lim\limits_{h\to0^+} 2 = 2$

Since $0 \ne 2$, the function is not differentiable at the point P.

3. Left-hand derivative:

$\lim\limits_{h\to0^-} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{\sqrt{1+h} - 1}{h}$

$= \lim\limits_{h\to0^-} \dfrac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)}$

$= \lim\limits_{h\to0^-} \dfrac{(1+h) - 1}{h(\sqrt{1+h} + 1)}$

$= \lim\limits_{h\to0^-} \dfrac{1}{\sqrt{1+h} + 1} = \dfrac{1}{2}$

Right-hand derivative:

$\lim\limits_{h\to0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^+} \dfrac{[2(1+h) - 1] - 1}{h}$

$= \lim\limits_{h\to0^+} \dfrac{2h}{h}$

$= \lim\limits_{h\to0^+} 2 = 2$

Since $\dfrac{1}{2} \ne 2$, the function is not differentiable at the point P.

4. Left-hand derivative:

$\lim\limits_{h\to0^-} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^-} \dfrac{(1+h) - 1}{h} = \lim\limits_{h\to0^-} 1 = 1$

Right-hand derivative:

$\lim\limits_{h\to0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to0^+} \dfrac{\dfrac{1}{1+h} - 1}{h}$

$= \lim\limits_{h\to0^+} \dfrac{1 - (1+h)}{h(1+h)}$

$= \lim\limits_{h\to0^+} \dfrac{-h}{h(1+h)}$

$= \lim\limits_{h\to0^+} -\dfrac{1}{1+h} = -1$

Since $1 \ne -1$, the function is not differentiable at the point P.

5. (a) All points in $[-3, 2]$

(b) None

(c) None

6. (a) All points in $[-2, 3]$

(b) None

(c) None

7. (a) All points in $[-3, 3]$ except $x = 0$

(b) None

(c) $x = 0$

8. (a) All points in $[-2, 3]$ except $x = -1$, 0, 2

(b) $x = -1$

(c) $x = 0$, $x = 2$

9. (a) All points in $[-1, 2]$ except $x = 0$

(b) $x = 0$

(c) None

10. (a) All points in $[-3, 3]$ except $x = -2$, 2

(b) $x = -2$, $x = 2$

(c) None

11. Since $\lim\limits_{x\to0} \tan^{-1} x = \tan^{-1} 0 = 0 \ne y(0)$, the problem is a discontinuity.

12. $\lim\limits_{h\to0^-} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h\to0^-} \dfrac{h^{4/5}}{h} = \lim\limits_{h\to0^-} \dfrac{1}{h^{1/5}} = -\infty$

$\lim\limits_{h\to0^+} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h\to0^+} \dfrac{h^{4/5}}{h} = \lim\limits_{h\to0^+} \dfrac{1}{h^{1/5}} = \infty$

The problem is a cusp.

13. Note that $y = x + \sqrt{x^2} + 2 = x + |x| + 2$

$= \begin{cases} 2, & x \le 0 \\ 2x + 2, & x > 0. \end{cases}$

$\lim\limits_{h\to0^-} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h\to0^-} \dfrac{2 - 2}{h} = \lim\limits_{h\to0^-} 0 = 0$

$\lim\limits_{h\to0^+} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h\to0^+} \dfrac{(2h+2) - 2}{h} = \lim\limits_{h\to0^+} 2 = 2$

The problem is a corner.

14. $\lim\limits_{h \to 0} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h \to 0} \dfrac{(3 - \sqrt[3]{h}) - 3}{h} = \lim\limits_{h \to 0} -\dfrac{\sqrt[3]{h}}{h}$

$= \lim\limits_{h \to 0}\left(-\dfrac{1}{h^{2/3}}\right) = -\infty$

The problem is a vertical tangent.

15. Note that $y = 3x - 2|x| - 1 = \begin{cases} 5x - 1, & x \le 0 \\ x - 1, & x > 0 \end{cases}$

$\lim\limits_{h \to 0^-} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{(5h - 1) - (-1)}{h} = \lim\limits_{h \to 0^-} 5 = 5$

$\lim\limits_{h \to 0^+} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{(h - 1) - (-1)}{h} = \lim\limits_{h \to 0^+} 1 = 1$

The problem is a corner.

16. $\lim\limits_{h \to 0^-} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{\sqrt[3]{|h|} - 0}{h} = \lim\limits_{h \to 0^-} \dfrac{-\sqrt[3]{h}}{h}$

$= \lim\limits_{h \to 0^-}\left(-\dfrac{1}{h^{2/3}}\right) = -\infty$

$\lim\limits_{h \to 0^+} \dfrac{y(0+h) - y(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{\sqrt[3]{|h|} - 0}{h} = \lim\limits_{h \to 0^+} \dfrac{-\sqrt[3]{h}}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{1}{h^{2/3}} = \infty$

The problem is a cusp.

17. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{4(0.001) - (0.001)^2 - (4(-0.001) - (-0.001)^2)}{0.002}$

$= 4$, yes it is differentiable.

18. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{4(3.001) - (3.001)^2 - (4(2.999) - (2.999)^2}{0.002}$

$= -2$, yes it is differentiable.

19. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{4(1.001) + (1.001)^2 - 4(0.999) - (0.99)^2}{0.002}$

$= 2$, yes it is differentiable.

20. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{(0.001)^3 - 4(0.001) - ((-0.001)^3 - 4(-0.001))}{0.002}$

$= -3.999999$, yes it is differentiable.

21. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{(-1.999)^3 - 4(-1.999) - ((-2.001)^3 - 4(-2.001))}{0.002}$

$= 8.000001$, yes it is differentiable.

22. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{h}$

$= \dfrac{(2.001)^3 - 4(2.001) - ((1.999)^3 - 4(1.999))}{0.002}$

$= -8.000001$, yes it is differentiable.

23. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{(0.001)^{2/3} - (-0.001)^{2/3}}{0.002}$

$= 0$, no it is not differentiable.

24. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{3.001 - 3 - (2.999 - 3)}{0.002}$

$= 0$, no it is not differentiable.

25. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$= \dfrac{(0.001)^{2/5} - (-0.001)^{2/5}}{0.002}$

$= 0$, no it is not differentiable.

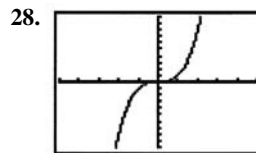**26.** $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{h}$

$= \dfrac{(0.001)^{4/5} - (-0.001)^{4/5}}{0.002}$

$= 0$, no it is not differentiable.

27.

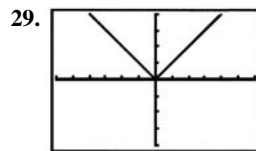[-2π, 2π] by [-1.5, 1.5]

$\dfrac{dy}{dx} = \sin x$

28.
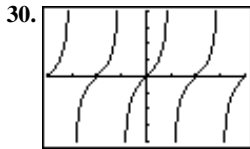

[−5, 5] by [−10, 10]

$\dfrac{dy}{dx} = x^3$

29.
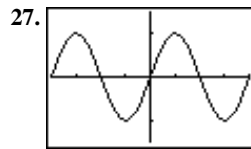
[−6, 6] by [−4, 4]

$\dfrac{dy}{dx} = \text{abs}(x) \text{ or } |x|$

30.

[−2π, 2π] by [−4, 4]

$\dfrac{dy}{dx} = \tan x$

Note: Due to the way NDER is defined, the graph of $y = $ NDER (x) actually has two asymptotes for each asymptote of $y = \tan x$. The asymptotes of

$y = $ NDER (x) occur at $x = \dfrac{\pi}{2} + k\pi \pm 0.001,$ where k is an integer. A good window for viewing this behavior is [1.566, 1.576] by [−1000, 1000].

31. Find the zeros of the denominator.

$x^2 - 4x - 5 = 0$
$(x+1)(x-5) = 0$
$x = -1 \text{ or } x = 5$

The function is a rational function, so it is differentiabe for all x in its domain: all reals except $x = -1, 5$.

32. The function is differentiable except possibly where $3x - 6 = 0,$ that is, at $x = 2.$ We check for differentiability at $x = 2,$ using k instead of the usual $h,$ in order to avoid confusion with the function $h(x).$

$\displaystyle\lim_{k\to 0} \frac{h(2+k) - h(2)}{k} = \lim_{k\to 0} \frac{[\sqrt[3]{3(2+k)-6}+5]-5}{k}$

$= \displaystyle\lim_{k\to 0} \frac{\sqrt[3]{3k}}{k} = \sqrt[3]{3}\lim_{k\to 0}\frac{1}{k^{2/3}} = \infty$

The function has a vertical tangent at $x = 2.$ It is differentiable for all reals except $x = 2.$

33. Note that the sine function is odd, so

$P(x) = \sin\left(|x|\right) - 1 = \begin{cases} -\sin x - 1, & x < 0 \\ \sin x - 1, & x \geq 0. \end{cases}$

The graph of $P(x)$ has a corner at $x = 0.$ The function is differentiable for all reals except $x = 0.$

34. Since the cosine function is even, so $Q(x) = 3\cos\left(|x|\right)$

$= 3\cos x.$ The function is differentiable for all reals.

35. The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3.$ Check $x = 0$:

$\displaystyle\lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)^2 - 1}{h} = \lim_{h\to 0^-} \frac{h^2 + 2h}{h}$

$= \displaystyle\lim_{h\to 0^-}(h+2) = 2$

$\displaystyle\lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^+} \frac{(2h+1)-1}{h} = \lim_{h\to 0^+} 2 = 2$

The function is differentiable at $x = 0.$

Check $x = 3$:

Since $g(3) = (4-3)^2 = 1$ and

$\displaystyle\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(2x+1) = 2(3)+1 = 7,$ the function is not continuous (and hence not differentiable) at $x = 3.$ The function is differentiable for all reals except $x = 3.$

36. Note that $C(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases},$ so it is differentiable for all x except possibly at $x = 0.$

Check $x = 0$:

$\displaystyle\lim_{h\to 0} \frac{C(0+h) - C(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0$

The function is differentiable for all reals.

37. The function $f(x)$ does not have the intermediate value property. Choose some a in $(-1, 0)$ and b in $(0, 1).$ Then $f(a) = 0$ and $f(b) = 1,$ but f does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, f cannot be the derivative of any function on $[-1, 1].$

38. (a) $x = 0$ is not in their domains, or, they are both discontinuous at $x = 0.$

(b) For $\dfrac{1}{x}$: NDER $\left(\dfrac{1}{x}, 0\right) = 1{,}000{,}000$

For $\dfrac{1}{x^2}$: NDER $\left(\dfrac{1}{x^2}, 0\right) = 0$

(c) It returns an incorrect response because even though these functions are not defined at $x = 0,$ they are defined at $x = \pm 0.001.$ The responses differ from each other

because $\dfrac{1}{x^2}$ is even (which automatically makes NDER

$\left(\dfrac{1}{x^2}, 0\right) = 0$) and $\dfrac{1}{x}$ is odd.

39. (a) $\displaystyle\lim_{x\to 1^-} f(x) = f(1)$

$\displaystyle\lim_{x\to 1^-}(3-x) = a(1)^2 + b(1)$

$2 = a + b$

The relationship is $a + b = 2.$

39. Continued

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{3 - (1+h) - 2}{h}$$
$$= \lim_{h \to 0^-} (-1) = -1$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{a(1+h)^2 + b(1+h) - 2}{h}$$
$$= \lim_{h \to 0^-} \frac{a + 2ah + ah^2 + b + bh - 2}{h}$$
$$= \lim_{h \to 0^-} \frac{2ah + ah^2 + bh + (a+b-2)}{h}$$
$$= \lim_{h \to 0^-} (2a + ah + b)$$
$$= 2a + b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$.
Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.
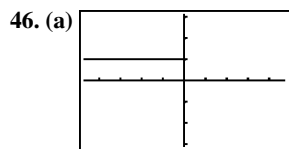
40. True. See Theorem 1.

41. False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

42. B.

43. A. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$

$$= \frac{\sqrt[3]{1.001 - 1} - \sqrt[3]{0.999 - 1}}{0.002}$$
$$= \infty$$
$$f(1) = \sqrt[3]{1 - 1} = 0$$

44. B. $\lim\limits_{h \to 0^-} \dfrac{2(0+h) + 1 - (2(0) + 1)}{h}$

$$= \lim_{h \to 0} \frac{2h}{h} = 2$$

45. C. $\lim\limits_{h \to 0} \dfrac{(0+h)^2 + 1 - (0^2 + 1)}{h}$

$$= \lim_{h \to 0} \frac{h^2}{h} = 0$$

46. (a)

[−4.7, 4.7] by [−3.1, 3.1]

(b) You can use Trace to help see that the value of Y1 is 1 for every $x < 0$ and is 0 for every $x \geq 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$

(c)
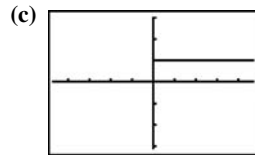
[−4.7, 4.7] by [−3.1, 3.1]

(d) You can use Trace to help see that the value of Y1 is 0 for every $x < 0$ and is 1 for every $x \geq 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$

47. (a)
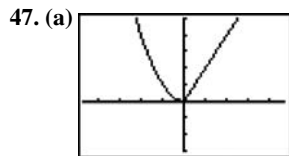
[−4.7, 4.7] by [−3, 5]

(b) See exersize 46.

(c)
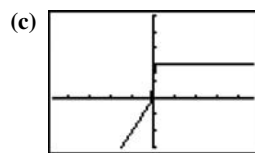
[−4.7, 4.7] by [−3, 5]

(d) NDER (Y1, x, −0.1) = −0.1, NDER(Y1, x, 0) = 0.9995, NDER (Y1, x, 0.1) = 2.

48. (a) Note that $-x \leq \sin\dfrac{1}{x} \leq x$, for all x, so $\lim\limits_{x \to 0}\left(x \sin\dfrac{1}{x}\right) = 0$ by the Sandwich Theorem. Therefore, f is continuous at $x = 0$.

(b) $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h \sin\dfrac{1}{h} - 0}{h} = \sin\dfrac{1}{h}$

(c) The limit does not exist because $\sin\dfrac{1}{h}$ oscillates between −1 and 1 an infinite number of times arbitrarily close to $h = 0$ (that is, for h in any open interval containing 0).

(d) No, because the limit in part (c) does not exist.

(e) $\dfrac{g(0+h) - g(0)}{h} = \dfrac{h^2 \sin\left(\dfrac{1}{h}\right) - 0}{h} = h \sin\dfrac{1}{h}$

As noted in part (a), the limit of this as x approaches zero is 0, so $g'(0) = 0$.

Section 3.3 Rules for Differentiation
(pp. 116–126)

Quick Review 3.3

1. $(x^2 - 2)(x^{-1} + 1) = x^2 x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
$$= x + x^2 - 2x^{-1} - 2$$

2. $\left(\dfrac{x}{x^2+1}\right)^{-1} = \dfrac{x^2+1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = x + x^{-1}$

3. $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2} = 3x^2 - 2x^{-1} + 5x^{-2}$

4. $\dfrac{3x^4 - 2x^3 + 4}{2x^2} = \dfrac{3x^4}{2x^2} - \dfrac{2x^3}{2x^2} + \dfrac{4}{2x^2}$

$= \dfrac{3}{2}x^2 - x + 2x^{-2}$

5. $(x^{-1}+2)(x^{-2}+1) = x^{-1}x^{-2} + x^{-1} \cdot 1 + 2x^{-2} + 2 \cdot 1$

$= x^{-3} + x^{-1} + 2x^{-2} + 2$

6. $\dfrac{x^{-1}+x^{-2}}{x^{-3}} = x^3(x^{-1}+x^{-2}) = x^2 + x$
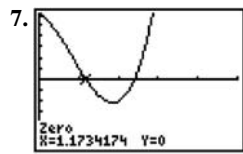
7.

Zero
X=1.1734174 Y=0

[0, 5] by [−6, 6]

At $x \approx 1.173$, $500x^6 \approx 1305$.

At $x \approx 2.394$, $500x^6 \approx 94{,}212$.

After rounding, we have:

At $x \approx 1$, $500x^6 \approx 1305$.

At $x \approx 2$, $500x^6 \approx 94{,}212$.

8. (a) $f(10) = 7$

(b) $f(0) = 7$

(c) $f(x+h) = 7$

(d) $\displaystyle\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a} = \lim_{x\to a}\dfrac{7-7}{x-a} = \lim_{x\to a} 0 = 0$

9. These are all constant functions, so the graph of each function is a horizontal line and the derivative of each function is 0.

10. (a) $f'(x) = \displaystyle\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\frac{x+h}{\pi}-\frac{x}{\pi}}{h}$

$= \displaystyle\lim_{h\to 0}\dfrac{x+h-x}{\pi h} = \lim_{h\to 0}\dfrac{1}{\pi} = \dfrac{1}{\pi}$

(b) $f'(x) = \displaystyle\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\frac{\pi}{x+h}-\frac{\pi}{x}}{h}$

$= \displaystyle\lim_{h\to 0}\dfrac{\pi x - \pi(x+h)}{hx(x+h)} = \lim_{h\to 0}\dfrac{-\pi h}{hx(x+h)}$

$= \displaystyle\lim_{h\to 0} -\dfrac{\pi}{x(x+h)} = -\dfrac{\pi}{x^2} = -\pi x^{-2}$

Section 3.3 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}(-x^2) + \dfrac{d}{dx}(3) = -2x + 0 = -2x$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) - \dfrac{d}{dx}(x) = x^2 - 1$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(1) = 2 + 0 = 2$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(x) + \dfrac{d}{dx}(1) = 2x + 1 + 0 = 2x + 1$

5. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) + \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right) + \dfrac{d}{dx}(x)$

$= x^2 + x + 1$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1) - \dfrac{d}{dx}(x) + \dfrac{d}{dx}(x^2) - \dfrac{d}{dx}(x^3)$

$= 0 - 1 + 2x - 3x^2 = -1 + 2x - 3x^2$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3 - 2x^2 + x + 1)$

$= 3x^2 - 4x + 1 = 0$

$x = \dfrac{1}{3},\ 1$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3 - 4x^2 + x + 2)$

$= 3x^2 - 8x + 1 = 0$

$x = \dfrac{4-\sqrt{13}}{3},\ \dfrac{4+\sqrt{13}}{3}$ or

$\approx 0.131,\ 2.535$

9. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 4x^2 + 1)$

$= 4x^3 - 8x = 0$

$x = 0,\ \pm\sqrt{2}$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4x^3 - 6x^2 - 1)$

$= 12x^2 - 12x = 0$

$x = 0,\ 1$

11. $\dfrac{dy}{dx} = \dfrac{d}{dx}(5x^3 - 3x^5)$

$= 15x^2 - 15x^4 = 0$

$x = -1,\ 0,\ 1$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 7x^3 + 2x^2 + 15)$

$= 4x^3 - 21x^2 + 4x = 0$

$x = 0,\ \dfrac{21-\sqrt{377}}{8} \approx 0.198,\ \dfrac{21+\sqrt{377}}{8} \approx 5.052$

13. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$= (x+1)\dfrac{d}{dx}(x^2+1) + (x^2+1)\dfrac{d}{dx}(x+1)$

$= (x+1)(2x) + (x^2+1)(1)$

$= 2x^2 + 2x + x^2 + 1$

$= 3x^2 + 2x + 1$

13. Continued

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$= \dfrac{d}{dx}(x^3+x^2+x+1)$

$= 3x^2+2x+1$

14. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right)$

$= \dfrac{x\dfrac{d}{dx}(x^2+3)-(x^2+3)\dfrac{d}{dx}(x)}{x^2}$

$= \dfrac{x(2x)-(x^2+3)}{x^2}$

$= \dfrac{x^2-3}{x^2}$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+3}{x}\right) = \dfrac{d}{dx}(x+3x^{-1}) = 1-3x^{-2}$

$= 1-\dfrac{3}{x^2}$

This is equivalent to the answer in part (a).

15. $(x^3+x+1)(x^4+x^2+1)$

$\dfrac{d}{dx}(x^7+2x^5+x^4+2x^3+x^2+x+1)$

$= 7x^6+10x^4+4x^3+6x^2+2x+1$

16. $(x^2+1)(x^3+1)$

$\dfrac{d}{dx}(x^5+x^3+x^2+1)$

$= 5x^4+3x^2+2x$

17. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2x+5}{3x-2} = \dfrac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = -\dfrac{19}{(3x-2)^2}$

18. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2+5x-1}{x^2}\right) = \dfrac{d}{dx}(1+5x^{-1}-x^{-2})$

$= 0-5x^{-2}+2x^{-3} = -\dfrac{5}{x^2}+\dfrac{2}{x^3}$

19. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{(x-1)(x^2+x+1)}{x^3}\right) = \dfrac{d}{dx}\left(\dfrac{x^3-1}{x^3}\right)$

$= \dfrac{d}{dx}(1-x^{-3}) = 0+3x^{-4} = \dfrac{3}{x^4}$

20. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1-x}{1+x^2}\right) = \dfrac{(1+x^2)(-1)-(1-x)(2x)}{(1+x^2)^2}$

$= \dfrac{x^2-2x-1}{(1+x^2)^2}$

21. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{1-x^3}\right) = \dfrac{(1-x^3)(2x)-x^2(-3x^2)}{(1-x^3)^2} = \dfrac{x^4+2x}{(1-x^3)^2}$

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{(x+1)(x+2)}{(x-1)(x-2)}\right) = \dfrac{d}{dx}\left(\dfrac{x^2+3x+2}{x^2-3x+2}\right)$

$= \dfrac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$

$= \dfrac{(2x^3-3x^2-5x+6)-(2x^3+3x^2-5x-6)}{(x^2-3x+2)^2}$

$= \dfrac{12-6x^2}{(x^2-3x+2)^2}$

23. (a) At $x=0$, $\dfrac{d}{dx}(uv) = u(0)v'(0)+v(0)\,u'(0)$

$= (5)(2)+(-1)(-3) = 13$

(b) At $x=0$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(0)u'(0)-u(0)v'(0)}{[v(0)]^2}$

$= \dfrac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$

(c) At $x=0$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{u(0)v'(0)-v(0)u'(0)}{[u(0)]^2}$

$= \dfrac{(5)(2)-(-1)(-3)}{(5)^2} = \dfrac{7}{25}$

(d) At $x=0$, $\dfrac{d}{dx}(7v-2u) = 7v'(0)-2u'(0)$

$= 7(2)-2(-3) = 20$

24. (a) At $x=2$, $\dfrac{d}{dx}(uv) = u(2)v'(2)+v(2)u'(2)$

$= (3)(2)+(1)(-4) = 2$

(b) At $x=2$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(2)u'(2)-u(2)v'(2)}{[v(2)]^2}$

$= \dfrac{(1)(-4)-(3)(2)}{(1)^2} = -10$

(c) At $x=2$, $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{u(2)v'(2)-v(2)u'(2)}{[u(2)]^2}$

$= \dfrac{(3)(2)-(1)(-4)}{(3)^2} = \dfrac{10}{9}$

(d) Use the result from part (a) for $\dfrac{d}{dx}(uv)$.

At $x=2$, $\dfrac{d}{dx}(3u-2v+2uv)$

$= 3u'(2)-2v'(2)+2\dfrac{d}{dx}(uv)$

$= 3(-4)-2(2)+2(2)$

$= -12$

25. $y'(x) = 2x+5$

$y'(3) = 2(3)+5 = 11$

The slope is 11. (iii)

26. The given equation is equivalent to $y = \dfrac{3}{2}x + 6$, so the slope

is $\dfrac{3}{2}$. (iii)

27. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^3 + 1}{2x}\right)$

$= \dfrac{(3x^2)2x - 2(x^3 + 1)}{4x^2}$

$= \dfrac{4x^3 - 2}{4x^2}$

$y'(1) = \dfrac{4(1)^3 - 2}{4(1)^2} = \dfrac{1}{2}$

$y(1) = \dfrac{(1)^3 + 1}{2(1)} = 1$

$y = \dfrac{1}{2}(x - 1) + 1 = \dfrac{1}{2}x + \dfrac{1}{2}$

28. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^4 + 2}{x^2}\right)$

$= \dfrac{(4x^3)x^2 - 2x(x^4 + 2)}{x^4}$

$= \dfrac{2x^4 - 4}{x^3}$

$y'(-1) = \dfrac{2(-1)^4 - 4}{(-1)^3} = 2$

$y(-1) = \dfrac{(-1)^4 + 2}{(-1)^2} = 3$

$y = 2(x + 1) + 3$

$y = 2x + 5$

29. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4x^{-2} - 8x + 1)$

$= -8x^{-3} - 8$

30. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^{-4}}{4} - \dfrac{x^{-3}}{3} + \dfrac{x^{-2}}{2} - x^{-1} + 3\right)$

$= -x^{-5} + x^{-4} - x^{-3} + x^{-2}$

31. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{\sqrt{x} - 1}{\sqrt{x} + 1} = \dfrac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$

32. $\dfrac{dy}{dx} = \dfrac{d}{dx}2\sqrt{x} - \dfrac{1}{\sqrt{x}}$

$= \dfrac{1}{\sqrt{x}} + \dfrac{1}{2x^{3/2}}$

33. $y \ = x^4 + x^3 - 2x^2 + x - 5$
$y^{\mathrm{I}} \ = 4x^3 + 3x^2 - 2x + 1$
$y^{\mathrm{II}} \ = 12x^2 + 6x - 2$
$y^{\mathrm{III}} = 24x + 6$
$y^{\mathrm{IV}} = 24$

34. $y \ = x^2 + x + 3$
$y^{\mathrm{I}} \ = 2x + 1$
$y^{\mathrm{II}} \ = 2$
$y^{\mathrm{III}} = 0$
$y^{\mathrm{IV}} = 0$

35. $y \ = x^{-1} + x^2$
$y^{\mathrm{I}} \ = -x^{-2} + 2x$
$y^{\mathrm{II}} \ = 2x^{-3} + 2$
$y^{\mathrm{III}} = -6x^{-4}$
$y^{\mathrm{IV}} = -24x^{-5}$

36. $y \ = \dfrac{x + 1}{x}$

$y^{\mathrm{I}} \ = \dfrac{1}{x^2}$

$y^{\mathrm{II}} \ = -\dfrac{2}{x^3}$

$y^{\mathrm{III}} = \dfrac{6}{x^4}$

$y^{\mathrm{IV}} = \dfrac{24}{x^5}$

37. $y'(x) = 3x^2 - 3$

$y'(2) = 3(2)^2 - 3 = 9$

The tangent line has slope 9, so the perpendicular line has

slope $-\dfrac{1}{9}$ and passes through (2, 3).

$y = -\dfrac{1}{9}(x - 2) + 3$

$y = -\dfrac{1}{9}x + \dfrac{29}{9}$

Graphical support:
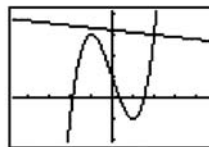
[−4.7, 4.7] by [−2.1, 4.1]

38. $y'(x) = 3x^2 + 1$

The slope is 4 when $3x^2 + 1 = 4$, at $x = \pm 1$. The tangent at
$x = -1$ has slope 4 and passes through (−1, −2), so its
equation is $y = 4(x + 1) - 2$, or $y = 4x + 2$. The tangent at
$x = 1$ has slope 4 and passes through (1, 2), so its equation is
$y = 4(x - 1) + 2$, or $y = 4x - 2$. The smallest slope occurs
when $3x^2 + 1$ is minimized, so the smallest slope is 1 and
occurs at $x = 0$.

Graphical support:
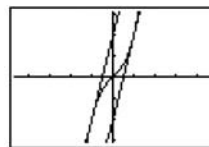
[−4.7, 4.7] by [−3.1, 3.1]

39. $y'(x) = 6x^2 - 6x - 12$
$\quad = 6(x^2 - x - 2)$
$\quad = 6(x + 1)(x - 2)$

The tangent is parallel to the x-axis when $y' = 0$, at $x = -1$ and at $x = 2$. Since $y(-1) = 27$ and $y(2) = 0$, the two points where this occurs are $(-1, 27)$ and $(2, 0)$.
Graphical support:

$[-3, 3]$ by $[-10, 30]$

40. $y'(x) = 3x^2$
$\quad y'(-2) = 12$

The tangent line has slope 12 and passes through $(-2, -8)$, so its equation is $y = 12(x + 2) - 8$, or $y = 12x + 16$. The x-intercept is $-\dfrac{4}{3}$ and the y-intercept is 16.

Graphical support:

$[-3, 3]$ by $[-20, 20]$

41. $y'(x) = \dfrac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2} = \dfrac{-4x^2 + 4}{(x^2 + 1)^2}$

At the origin: $y'(0) = 4$
The tangent is $y = 4x$.
At $(1, 2)$: $y'(1) = 0$
The tangent is $y = 2$.
Graphical support:
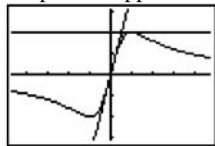
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

42. $y'(x) = \dfrac{(4 + x^2)(0) - 8(2x)}{(4 + x^2)^2} = -\dfrac{16x}{(4 + x^2)^2}$

$y'(2) = -\dfrac{1}{2}$

The tangent has slope $-\dfrac{1}{2}$ and passes through $(2, 1)$. Its

equation is $y = -\dfrac{1}{2}(x - 2) + 1$, or $y = -\dfrac{1}{2}x + 2$.
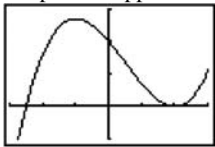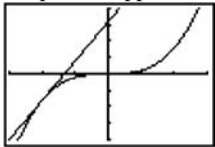
Graphical support:
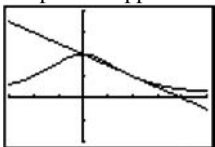
$[-3, 5]$ by $[-2, 4]$

43. (a) Let $f(x) = x$.

$\dfrac{d}{dx}(x) = f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{(x+h) - x}{h}$

$= \lim_{h \to 0} \dfrac{h}{h} = \lim_{h \to 0}(1) = 1$

(b) Note that $u = u(x)$ is a function of x.

$\dfrac{d}{dx}(-u) = \lim_{h \to 0} \dfrac{-u(x+h) - [-u(x)]}{h}$

$= \lim_{h \to 0} \left(-\dfrac{u(x+h) - u(x)}{h} \right)$

$= -\lim_{h \to 0} \dfrac{u(x+h) - u(x)}{h} = -\dfrac{du}{dx}$

44. $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot \dfrac{d}{dx} f(x) + f(x) \cdot \dfrac{d}{dx}(c)$

$= c \cdot \dfrac{d}{dx} f(x) + 0 = c \cdot \dfrac{d}{dx} f(x)$

45. $\dfrac{d}{dx}\left(\dfrac{1}{f(x)} \right) = \dfrac{f(x) \cdot 0 - 1 \cdot \dfrac{d}{dx} f(x)}{[f(x)]^2} = -\dfrac{f'(x)}{[f(x)]^2}$

46. $\dfrac{dP}{dV} = \dfrac{d}{dV}\left(\dfrac{nRT}{V - nb} - \dfrac{an^2}{V^2} \right)$

$= \dfrac{(V - nb)\dfrac{d}{dV}(nRT) - (nRT)\dfrac{d}{dV}(V - nb)}{(V - nb)^2} - \dfrac{d}{dV}(an^2 V^{-2})$

$= \dfrac{0 - nRT}{(V - nb)^2} + 2an^2 V^{-3}$

$= -\dfrac{nRT}{(V - nb)^2} + \dfrac{2an^2}{V^3}$

47. $\dfrac{ds}{dt} = \dfrac{d}{dt}(4.9t^2) = 9.8t$

$\dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}(9.8t) = 9.8$

48. $\dfrac{dR}{dM} = \dfrac{d}{dM}\left[M^2\left(\dfrac{C}{2} - \dfrac{M}{3} \right) \right]$

$= \dfrac{d}{dM}\left(\dfrac{C}{2}M^2 - \dfrac{1}{3}M^3 \right)$

$= CM - M^2$

49. If the radius of a circle is changed by a very small amount Δr, the change in the area can be thought of as a very thin strip with length given by the circumference, $2\pi r$, and width Δr. Therefore, the change in the area can be thought of as $(2\pi r)(\Delta r)$, which means that the change in the area divided by the change in the radius is just $2\pi r$.

50. If the radius of a sphere is changed by a very small amount Δr, the change in the volume can be thought of as a very thin layer with an area given by the surface area, $4\pi r^2$, and a thickness given by Δr. Therefore, the change in the volume can be thought of as $(4\pi r^2)(\Delta r)$, which means that the change in the volume divided by the change in the radius is just $4\pi r^2$.

51. Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree x years from now. Then $t(0) = 156$, $y(0) = 12$, $t'(0) = 13$, and $y'(0) = 1.5$. The rate of increase of production is

$$\frac{d}{dx}(ty) = t(0)y'(0) + y(0)t'(0) = (156)(1.5) + (12)(13) = 390$$

bushels of annual production per year.

52. Let $m(x)$ be the number of members and $c(x)$ be the pavillion cost x years from now. Then $m(0) = 65$, $c(0) = 250$, $m'(0) = 6$, and $c'(0) = 10$. The rate of change of each member's share is $\dfrac{d}{dx}\left(\dfrac{c}{m}\right) = \dfrac{m(0)c'(0) - c(0)m'(0)}{[m(0)]^2}$

$$= \frac{(65)(10) - (250)(6)}{(65)^2} \approx -0.201 \text{ dollars per year. Each}$$

member's share of the cost is decreasing by approximately 20 cents per year.

53. False. π is a constant so $\dfrac{d}{dx}(\pi) = 0$.

54. True. $f'(x) = -\dfrac{1}{x^2}$ is never zero, so there are no horizontal tangents.

55. B. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$$= (2)(1) + (-1)(3)$$
$$= -1$$

56. D. $f(x) = x - \dfrac{1}{x}$
$$f'(x) = 1 + \frac{1}{x^2}$$
$$f''(x) = -\frac{2}{x^3}$$

57. E. $\dfrac{d}{dx}\left(\dfrac{x+1}{x^2-1}\right)$
$$= \frac{(x-1) - (x+1)}{(x-1)^2}$$
$$= \frac{-2}{(x-1)^2}$$

58. B. $f(x) = (x^2 - 1)(x^2 + 1)$
$$x = \pm 1$$

59. (a) It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

(b) It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

(c) $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$. This is equivalent to the product rule given in the text.

(d) Because dx is "infinitely small," and this could be thought of as dividing by zero.

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{u+du}{v+dv} - \dfrac{u}{v}$
$$= \frac{(u+du)(v) - (u)(v+dv)}{(v+dv)(v)}$$
$$= \frac{uv + vdu - uv - udv}{v^2 + vdv}$$
$$= \frac{vdu - udv}{v^2}$$

Quick Quiz Sections 3.1–3.3

1. D.

2. A. Since $\dfrac{dy}{dx}$ gives the slope:
$$m_1 = \frac{1-2}{-1-1} = \frac{1}{2}$$
$$m_2 = -\frac{1}{m_1} = -2$$

3. C. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{4x-3}{2x+1}$
$$= \frac{4(2x+1) - 2(4x-3)}{(2x-1)^2}$$
$$= \frac{10}{(2x-1)^2}$$

4. (a) $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 4x^2)$
$$= 4x^2 - 8x = 0$$
$$x = 0, \ \pm\sqrt{2}$$

(b) $x = 1 \qquad y = (1)^4 - 4(1)^2$
$$y = -3$$
$$y = m(x - x_1) + y_1$$
$$y = -4(x - 1) - 3$$
$$y = -4x + 1$$

4. Continued

(c) $m_2 = -\dfrac{1}{m_1} = \dfrac{1}{4}$

$$y = \frac{1}{4}(x-1) - 3$$
$$= \frac{1}{4}x - \frac{1}{4} - 3 = \frac{1}{4}x - \frac{13}{4}$$

Section 3.4 Velocity and Other Rates of Change (pp. 127–140)

Exploration 1   Growth Rings on a Tree

1. Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.

2. Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.

3. Since change in area is constant, so also is $\dfrac{\text{change in area}}{2\pi}$.

If we denote this latter constant by k, we have

$\dfrac{k}{\text{change in radius}} = r$, which means that r varies inversely

as the change in the radius. In other words, the change in radius must get smaller when r gets bigger, and vice-versa.

Exploration 2   Modeling Horizontal Motion

1. The particle reverses direction at about $t = 0.61$ and $t = 2.06$.
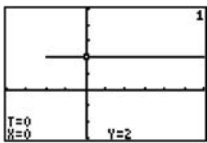
2. When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.
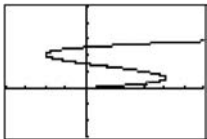
3. When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of t for when the particle reverses direction.
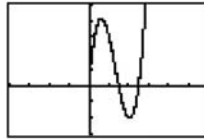
4. We can represent the velocity by graphing the parametric equations

$x_4(t) = x_1{}'(t) = 12t^2 - 32t + 15, \ y_4(t) = 2$ (part 1),

$x_5(t) = x_1{}'(t) = 12t^2 - 32t + 15, \ y_5(t) = t$ (part 2),

$x_6(t) = t, \ y_6(t) = x_1{}'(t) = 12t^2 - 32t + 15$ (part 3)
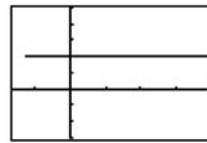
$[-8, 20]$ by $[-3, 5]$
(x_4, y_4)

$[-8, 20]$ by $[-3, 5]$
(x_5, y_5)
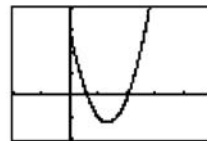
$[-2, 5]$ by $[-10, 20]$
(x_6, y_6)

For (x_4, y_4) and (x_5, y_5), the particle is moving to the right when the x-coordinate of the graph (velocity) is positive, moving to the left when the x-coordinate of the graph (velocity) is negative, and is stopped when the x-coordinate of the graph (velocity) is 0. For (x_6, y_6),the particle is moving to the right when the y-coordinate of the graph (velocity) is positive, moving to the left when the y-coordinate of the graph (velocity) is negative, and is stopped when the y-coordinate of the graph (velocity) is 0.

Exploration 3   Seeing Motion on a Graphing Calculator

1. Let tMin = 0 and tMax = 10.

2. Since the rock achieves a maximum height of 400 feet, set yMax to be slightly greater than 400, for example yMax = 420.

4. The grapher proceeds with constant increments of t (time), so pixels appear on the screen at regular time intervals. When the rock is moving more slowly, the pixels appear closer together. When the rock is moving faster, the pixels appear farther apart. We observe faster motion when the pixels are farther apart.

Quick Review 3.4

1. The coefficient of x^2 is negative, so the parabola opens downward.

Graphical support:
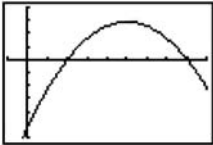
[−1, 9] by [−300, 200]

2. The y-intercept is $f(0) = -256$.

See the solution to Exercise 1 for graphical support.

3. The x-intercepts occur when $f(x) = 0$.

$-16x^2 + 160x - 256 = 0$
$-16(x^2 - 10x + 16) = 0$
$-16(x - 2)(x - 8) = 0$
$x = 2 \text{ or } x = 8$

The x-intercepts are 2 and 8. See the solution to Exercise 1 for graphical support.

4. Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$, the range is $(-\infty, 144]$.

See the solution to Exercise 1 for graphical support.

5. Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$, the vertex is at $(5, 144)$. See the solution to Exercise 1 for graphical support.

6. $f(x) = 80$
$-16x^2 + 160x - 256 = 80$
$-16x^2 + 160x - 336 = 0$
$-16(x^2 - 10x + 21) = 0$
$-16(x - 3)(x - 7) = 0$
$x = 3 \text{ or } x = 7$
$f(x) = 80$ at $x = 3$ and at $x = 7$.

See the solution to Exercise 1 for graphical support.

7. $\dfrac{dy}{dx} = 100$
$-32x + 160 = 100$
$60 = 32x$
$x = \dfrac{15}{8}$
$\dfrac{dy}{dx} = 100$ at $x = \dfrac{15}{8}$

Graphical support: the graph of NDER $f(x)$ is shown.
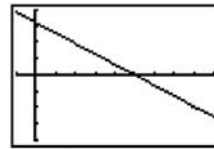
[−1, 9] by [−200, 200]

8. $\dfrac{dy}{dx} > 0$
$-32x + 160 > 0$
$-32x > -160$
$x < 5$

$\dfrac{dy}{dx} > 0$ when $x < 5$.

See the solution to Exercise 7 for graphical support.

9. Note that $f'(x) = -32x + 160$.

$\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h} = f'(3) = -32(3) + 160 = 64$

For graphical support, use the graph shown in the solution to Exercise 7 and observe that NDER $(f(x),\ 3) \approx 64$.

10. $f'(x) = -32x + 160$
$f''(x) = -32$

At $x = 7$ (and, in fact, at any other of x),

$\dfrac{d^2y}{dx^2} = -32.$

Graphical support: the graph of NDER(NDER $f(x)$) is shown.
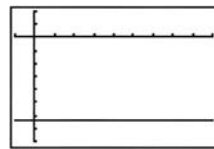
[−1, 9] by [−40, 10]

Section 3.4 Exercises

1. (a) $V(s) = s^3$

(b) $\dfrac{dv}{ds} = 3s^2$

(c) $V'(1) = 3(1)^2 = 3$
    $V'(5) = 3(5)^2 = 75$

(d) $\dfrac{\text{in}^3}{\text{in}}.$

2. (a) $A(C)\dfrac{C^2}{4\pi}$

(b) $\dfrac{dC}{dA} = \dfrac{2C}{4\pi} = \dfrac{C}{2\pi}$

2. Continued

(c) $A'(\pi) = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

$A'(6\pi) = \dfrac{6\pi}{2\pi} = 3$

(d) $\dfrac{in^2}{in}$.

3. (a) $A(s) = \dfrac{\sqrt{3}}{4}s^2$

(b) $\dfrac{dA}{ds} = \dfrac{\sqrt{3}}{2}s$

(c) $A'(2) = \dfrac{\sqrt{3}}{2}(2) = \sqrt{3}$

$A'(10) = \dfrac{\sqrt{3}}{2}(10) = 5\sqrt{3}$

(d) $\dfrac{in^2}{in}$.

4. (a) $A(r) = 2r^2$

(b) $\dfrac{dA}{dr} = 4r$

(c) $A'(1) = 4(1) = 4$
$A'(8) = 4(8) = 32$

(d) $\dfrac{in^2}{in}$.

5. (a) $s\,(ft)$

(b) $s'(1) = 18$, $s'(2.5) = 0$, $s'(3.5) = -12$

6. (a) $s\,(ft)$
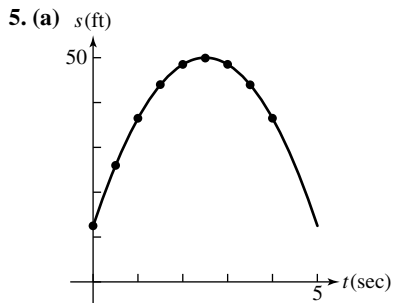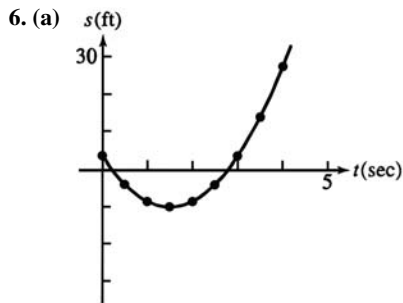
(b) $s'(1) = -6$, $s'(2.5) = 12$, $s'(3.5) = 24$

7. (a) We estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (days)	0	10	20	30	40	50
Slope (flies/ day)	0.5	3.0	13.0	14.0	3.5	0.5
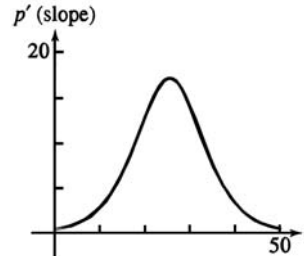
Horizontal axis: Days
Vertical axis: Flies per day

(b) Fastest: Around the 25th day
Slowest: Day 50 or day 0

8. $Q(t) = 200(30 - t)^2 = 200(900 - 60t + t^2)$
$= 180,000 - 12,000t + 200t^2$
$Q'(t) = -12,000 + 400t$
The rate of change of the amount of water in the tank after 10 minutes is $Q'(10) = -8000$ gallons per minute.
Note that $Q'(10) < 0$, so the rate at which the water is running *out* is positive. The water is running out at the rate of 8000 gallons per minute.

The average rate for the first 10 minutes is
$\dfrac{Q(10) - Q(0)}{10 - 0} = \dfrac{80,000 - 180,000}{10} = -10,000$ gallons per minute.
The water is flowing out at an average rate of 10,000 gallons per minute over the first 10 min.

9. (a) The particle moves forward when $v > 0$, for $0 \le t < 1$ and for $5 < t < 7$.
The particle moves backward when $v < 0$, for $1 < t < 5$.
The particle speeds up when v is negative and decreasing, for $1 < t < 2$, and when v is positive and increasing, for $5 < t < 6$.
The particle slows down when v is positive and decreasing, for $0 \le t < 1$ and for $6 < t < 7$, and when v is negative and increasing, for $3 < t < 5$.

(b) Note that the acceleration $a = \dfrac{dv}{dt}$ is undefined at $t = 2$, $t = 3$, and $t = 6$.
The acceleration is positive when v is increasing, for $3 < t < 6$.
The acceleration is negative when v is decreasing, for $0 \le t < 2$ and for $6 < t < 7$.
The acceleration is zero when v is constant, for $2 < t < 3$ and for $7 < t \le 9$.

(c) The particle moves at its greatest speed when $|v|$ is maximized, at $t = 0$ and for $2 < t < 3$.

9. Continued

(d) The particle stands still for more than an instant when v stays at zero, for $7 < t \le 9$.

10. (a) The particle is moving left when the graph of s has negative slope, for $2 < t < 3$ and for $5 < t \le 6$.
The particle is moving right when the graph of s has positive slope, for $0 \le t < 1$.
The particle is standing still when the graph of s is horizontal, for $1 < t < 2$ and for $3 < t < 5$.

(b) For $0 \le t < 1$: $v = \dfrac{2-0}{1-0} = 2$ cm/sec

$\text{Speed} = |v| = 2$ cm/sec

For $1 < t < 2$: $v = \dfrac{2-2}{2-1} = 0$ cm/sec

$\text{Speed} = |v| = 0$ cm/sec

For $2 < t < 3$: $v = \dfrac{-2-2}{3-2} = -4$ cm/sec

$\text{Speed} = |v| = 4$ cm/sec

For $3 < t < 5$: $v = \dfrac{-2-(-2)}{5-3} = 0$ cm/sec

$\text{Speed} = |v| = 0$ cm/sec

For $5 < t \le 6$: $v = \dfrac{-4-(-2)}{6-5} = -2$ cm/sec

$\text{Speed} = |v| = 2$ cm/sec

Velocity graph:

$v(t)$ (cm/sec)

Speed graph:

$|v(t)|$ (cm/sec)
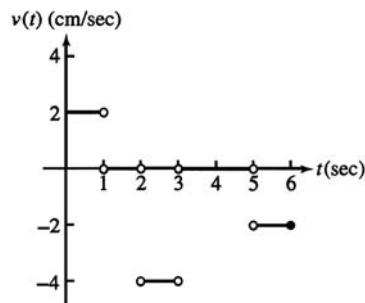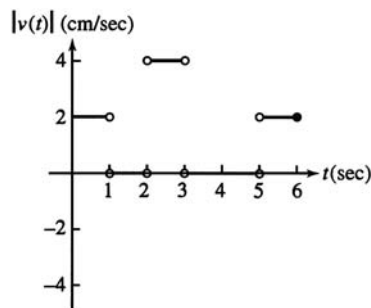
11. (a) The body reverses direction when v changes sign, at $t = 2$ and at $t = 7$.

(b) The body is moving at a constant speed, $|v| = 3$ m/sec, between $t = 3$ and $t = 6$.

(c) The speed graph is obtained by reflecting the negative portion of the velocity graph, $2 < t < 7$, about the x-axis.

Speed(m/sec)
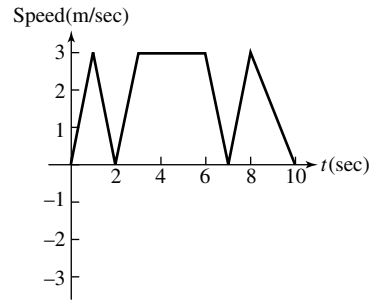
(d) For $0 \le t < 1$: $a = \dfrac{3-0}{1-0} = 3$ m/sec^2

For $1 < t < 3$: $a = \dfrac{-3-3}{3-1} = -3$ m/sec^2

For $3 < t < 6$: $a = \dfrac{-3-(-3)}{6-3} = 0$ m/sec^2

For $6 < t < 8$: $a = \dfrac{3-(-3)}{8-6} = 3$ m/sec^2

For $8 < t \le 10$: $a = \dfrac{0-3}{10-8} = -1.5$ m/sec^2

Acceleration (m/sec^2)
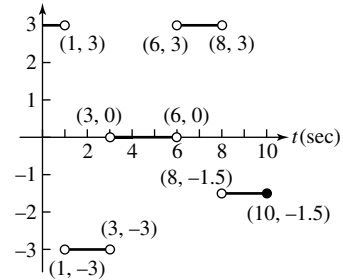
12. (a) It takes 135 seconds.

(b) Average speed $= \dfrac{\Delta F}{\Delta t} = \dfrac{5-0}{73-0} = \dfrac{5}{73}$

≈ 0.068 furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately

$\dfrac{\Delta F}{\Delta t} = \dfrac{4-2}{59-33} = \dfrac{2}{26} = \dfrac{1}{13} \approx 0.077$ furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

13. (a) Velocity: $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(24t - 0.8t^2) = 24 - 1.6t$ m/sec

Acceleration: $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(24 - 1.6t) = -1.6$ m/sec^2

13. Continued

(b) The rock reaches its highest point when
$v(t) = 24 - 1.6t = 0$, at $t = 15$. It took 15 seconds.

(c) The maximum height was $s(15) = 180$ meters.

(d) $s(t) = \dfrac{1}{2}(180)$

$24t - 0.8t^2 = 90$

$0 = 0.8t^2 - 24t + 90$

$t = \dfrac{24 \pm \sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$

$\approx 4.393,\ 25.607$

It took about 4.393 seconds to reach half its maximum height.

(e) $s(t) = 0$

$24t - 0.8t^2 = 0$

$0.8t(30 - t) = 0$

$t = 0$ or $t = 30$

The rock was aloft from $t = 0$ to $t = 30$, so it was aloft for 30 seconds.

14. On Mars:

Velocity $= \dfrac{ds}{dt} = \dfrac{d}{dt}(1.86t^2) = 3.72t$

Solving $3.72t = 16.6$, the downward velocity reaches 16.6 m/sec after about 4.462 sec.

On Jupiter:

Velocity $= \dfrac{ds}{dt} = \dfrac{d}{dt}(11.44t^2) = 22.88t$

Solving $22.88t = 16.6$, the downward velocity reaches 16.6 m/sec after about 0.726 sec.

15. The rock reaches its maximum height when the velocity $s'(t) = 24 - 9.8t = 0$, at $t \approx 2.449$. Its maximum height is about $s(2.449) \approx 29.388$ meters.

16. Moon:

$s(t) = 0$

$832t - 2.6t^2 = 0$

$2.6t(320 - t) = 0$

$t = 0$ or $t = 320$

It takes 320 seconds to return.

Earth:

$s(t) = 0$

$832t - 16t^2 = 0$

$16t(52 - t) = 0$

$t = 0$ or $t = 52$

It takes 52 seconds to return.

17. The following is one way to simulate the problem situation.

For the moon:

$x_1(t) = 3(t < 160) + 3.1(t \geq 160)$

$y_1(t) = 832t - 2.6t^2$

t-values: 0 to 320

window: [0, 6] by [−10,000, 70,000]

For the earth:

$x_1(t) = 3(t < 26) + 3.1(t \geq 26)$

$y_1(t) = 832t - 16t^2$

t-values: 0 to 52

window: [0, 6] by [−1000, 11,000]

18. (a) 190 ft/sec

(b) 2 seconds

(c) After 8 seconds, and its velocity was 0 ft/sec then

(d) After about 11 seconds, and it was falling 90 ft/sec then

(e) About 3 seconds (from the rocket's highest point)

(f) The acceleration was greatest just before the engine stopped. The acceleration was constant from $t = 2$ to $t = 11$, while the rocket was in free fall.

19. (a) Displacement: $= s(5) - s(0) = 12 - 2 = 10$ m

(b) Average velocity $= \dfrac{10 \text{ m}}{5 \text{ sec}} = 2$ m/sec

(c) Velocity $= s'(t) = 2t - 3$

At $t = 4$, velocity $= s'(4) = 2(4) - 3 = 5$ m/sec

(d) Acceleration $= s''(t) = 2$ m/sec^2

(e) The particle changes direction when

$s'(t) = 2t - 3 = 0$, so $t = \dfrac{3}{2}$ sec.

(f) Since the acceleration is always positive, the position s is at a minimum when the particle changes direction, at $t = \dfrac{3}{2}$ sec. Its position at this time is $s\left(\dfrac{3}{2}\right) = -\dfrac{1}{4}$ m.

20. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(-t^3 + 7t^2 - 14t + 8)$

$v(t) = -3t^2 + 14t - 14$

(b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(-3t^2 + 14t - 14)$

$a(t) = -6t + 14$

(c) $v(t) = -3t^2 + 14t - 14 = 0$

$t \approx 1.451,\ 3.215$

(d) The particle starts at the point $s = 0$ and moves left until it stops at $s = -0.631$ at $t = 1.451$, then it moves right to the point $s = 2.213$ at $t = 3.215$ where it stops again, and finally continues left from there on.

21. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(t - 2)^2(t - 4)$

$= (t - 2)(3t - 10)$

(b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(t - 2)(3t - 10)$

$a(t) = 6t - 16$

21. Continued

  (c) $v(t) = (t-2)(3t-10) = 0$

   $t = 2, \dfrac{10}{3}$

  (d) The particle starts at the point $s = -16$ when $t = 0$ and move right until it stops at $s = 0$ when $t = 2$, then it moves left to the point $s = -1.185$ when $t = \dfrac{10}{3}$ where it stops again, and finally continues right from there on.

22. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(t^3 - 6t^2 + 8t + 2)$

   $v(t) = 3t^2 - 12t + 8$

  (b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(3t^2 - 12t + 8)$

   $a(t) = 6t - 12$

  (c) $v(t) = 3t^2 - 12t + 8 = 0$

   $t \approx 0.845, \; 3.155$

  (d) The particle starts at the point $s = -16$ when $t = 0$ and moves right until it stops at $s = 5.079$ at $t = 0.845$, then it moves left to the point $s = -1.079$ at $t = 3.155$ where it stops again, and finally continues right from there on.

23. $v(t) = s'(t) = 3t^2 - 12t + 9$

   $a(t) = v'(t) = 6t - 12$

   Find when velocity is zero.

   $3t^2 - 12t + 9 = 0$

   $3(t^2 - 4t + 3) = 0$

   $3(t-1)(t-3) = 0$

   $t = 1 \text{ or } t = 3$

   At $t = 1$, the acceleration is $a(1) = -6$ m/sec^2

   At $t = 3$, the acceleration is $a(3) = 6$ m/sec^2

24. $a(t) = v'(t) = 6t^2 - 18t + 12$

   Find when acceleration is zero.

   $6t^2 - 18t + 12 = 0$

   $6(t^2 - 3t + 2) = 0$

   $6(t-1)(t-2) = 0$

   $t = 1 \text{ or } t = 2$

   At $t = 1$, the speed is $|v(1)| = |0| = 0$ m/sec.

   At $t = 2$, the speed is $|v(2)| = |-1| = 1$ m/sec.

25. (a) $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{12}\right)^2\right]$

   $= \dfrac{d}{dt}\left[6\left(1 - \dfrac{t}{6} + \dfrac{t^2}{144}\right)\right]$

   $= \dfrac{d}{dt}\left(6 - t + \dfrac{1}{24}t^2\right)$

   $= 0 - 1 + \dfrac{t}{12} = \dfrac{t}{12} - 1$

(b) The fluid level is falling fastest when $\dfrac{dy}{dt}$ is the most negative, at $t = 0$, when $\dfrac{dy}{dt} = -1$. The fluid level is falling slowest at $t = 12$, when $\dfrac{dy}{dt} = 0$.

(c)

y is decreasing and $\dfrac{dy}{dt}$ is negative over the entire interval y decreases more rapidly early in the interval, and the magnitude of $\dfrac{dy}{dt}$ is larger then. $\dfrac{dy}{dt}$ is 0 at $t = 12$, where the graph of y seems to have a horizontal tangent.

26. (a) To graph the velocity, we estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
v (km/hour)	0	56	75	56	0	−94	−225
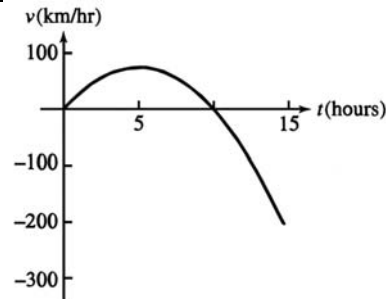
To graph the acceleration, we estimate the slope of the velocity grapy at several points as follows, and then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
a (km/hour^2)	30	15	0	−15	−30	−45	−60
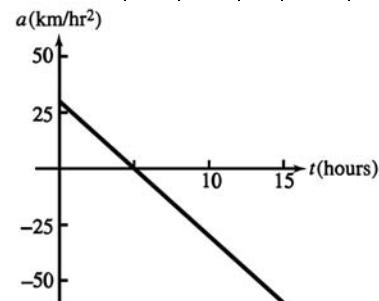
26. Continued

(b) $\dfrac{ds}{dt} = 30t - 3t^2$
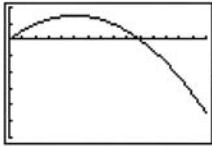
[0, 15] by [−300, 100]

$\dfrac{d^2s}{dt^2} = 30 - 6t$
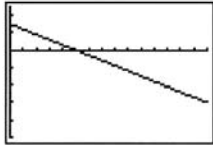
[0, 15] by [−100, 50]

The graphs are very similar.

27. (a) Average cost $= \dfrac{c(100)}{100} = \dfrac{11{,}000}{100} = \$110$ per machine

(b) $c'(x) = 100 - 0.2x$

Marginal cost $= c'(100) = \$80$ per machine

(c) Actual cost of 101st machine is
$c(101) - c(100) = \$79.90$, which is very close to the marginal cost calculated in part (b).

28. (a)
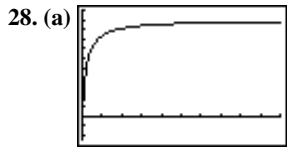
[0, 50] by [−500, 2200]

The values of x which make sense are the whole numbers, $x \geq 0$.

(b) Marginal revenue $= r'(x) = \dfrac{d}{dx}\left[2000\left(1 - \dfrac{1}{x+1}\right) \right]$

$= \dfrac{d}{dx}\left(2000 - \dfrac{2000}{x+1} \right)$

$= 0 - \dfrac{(x+1)(0) - (2000)(1)}{(x+1)^2} = \dfrac{2000}{(x+1)^2}$

(c) $r'(5) = \dfrac{2000}{(5+1)^2} = \dfrac{2000}{36} \approx 55.56$

The increase in revenue is approximately $55.56.

(d) The limit is 0. This means that as x gets large, one reaches a point where very little extra revenue can be expected from selling more desks.

29. (a)
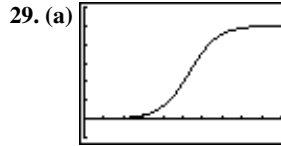
[0, 200] by [−2, 12]

(b) The values of x which make sense are the whole numbers, $x \geq 0$.

(c)
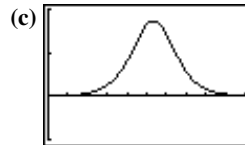
[0, 200] by [−0.1, 0.2]

P is most sensitive to changes in x when $\left| P'(x) \right|$ is largest. It is relatively sensitive to changes in x between approximately $x = 60$ and $x = 160$.

(d) The marginal profit, $P'(x)$, is greatest at $x = 106.44$. Since x must be an integer,
$P(106) \approx 4.924$ thousand dollars or \$4924.

(e) $P'(50) \approx 0.013$, or \$13 per package sold

$P'(100) \approx 0.165$, or \$165 per package sold

$P'(125) \approx 0.118$, or \$118 per package sold

$P'(150) \approx 0.031$, or \$31 per package sold

$P'(175) \approx 0.006$, or \$6 per package sold

$P'(300) \approx 10^{-6}$, or \$0.001 per package sold

(f) The limit is 10. The maximum possible profit is \$10,000 monthly.

(g) Yes. In order to sell more and more packages, the company might need to lower the price to a point where they won't make any additional profit.

30. The particle is at (5, 2) when $4t^3 - 16t^2 + 15t = 5$, which occurs at $t \approx 2.83$.

31. Graph C is position, graph A is velocity, and graph B is acceleration.
A is the derivative of C because it is positive, negative, and zero where C is increasing, decreasing, and has horizontal tangents, respectively. The relationship between B and A is similar.

32. Graph C is position, graph B is velocity, and graph A is acceleration.
B is the derivative of C because it is negative and zero where C is decreasing and has horizontal tangents, respectively.
A is the derivative of B because it is positive, negative, and zero where B is increasing, decreasing, and has horizontal tangents, respectively.

33. Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:
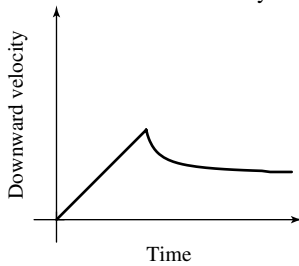
34. (a) $\dfrac{dV}{dr} = \dfrac{d}{dr}\left(\dfrac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\dfrac{dv}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of volume per foot of radius.

(b) The increase in the volume is

$\dfrac{4}{3}\pi(2.2)^3 - \dfrac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

35. Let v_0 be the exit velocity of a particle of lava. Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is

$\dfrac{ds}{dt} = v_0 - 32t$. Solving $\dfrac{ds}{dt} = 0$ gives $t = \dfrac{v_0}{32}$. Then the maximum height, in feet, is

$s\left(\dfrac{v_0}{32}\right) = v_0\left(\dfrac{v_0}{32}\right) - 16\left(\dfrac{v_0}{32}\right)^2 = \dfrac{v_0^2}{64}$. Solving

$\dfrac{v_0^2}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was

about 348.712 ft/sec. Multiplying by $\dfrac{3600 \text{ sec}}{1 \text{ h}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}}$,

we find that this is equivalent to about 237.758 mi/h.

36. By estimating the slope of the velocity graph at that point.

37. The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in $[-6, 8]$ by $[-3, 5]$.

(a) It begins at the point $(-5, 2)$ moving in the positive direction. After a little more than one second, it has moved a bit past $(6, 2)$ and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near $(-2, 2)$ and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

(b) The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \le t \le 2.167$ and $t \ge 3.180$. It slows down during the approximate intervals $0 \le t \le 1.153$ and $2.167 \le t \le 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed, $|\text{NDER } x(t)|$, using function mode in the window $[0, 5]$ by $[0, 10]$.

(c) The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(d) The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(e) The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase.
The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

(f) The particle is at $(5, 2)$ when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

38. (a) Solving $160 = 490t^2$ gives $t = \pm\dfrac{4}{7}$. It took $\dfrac{4}{7}$ of a second.

The average velocity was $\dfrac{160 \text{ cm}}{\left(\dfrac{4}{7}\right)}$ sec $= 280$ cm/sec.

39. Since profit = revenue – cost, the Sum and Difference Rule

gives $\dfrac{d}{dx}(\text{profit}) = \dfrac{d}{dx}(\text{revenue}) - \dfrac{d}{dx}(\text{cost})$, where x is the

number of units produced. This means that marginal profit = marginal revenue – marginal cost.

40. False. It is the absolute value of the velocity.

41. True. The acceleration is the first derivative of the velocity which, in turn, is the second derivative of the position function.

42. C. $\dfrac{d}{dx}\dfrac{x^2 - 2}{x + 4} = \dfrac{(x+4)(2x) - (x^2 - 2)}{x + 4}$

$f'(x) = \dfrac{x^2 + 8x - 2}{x + 4}$

$f'(-1) = \dfrac{(-1)^2 + 8(-1) - 2}{(-1) + 4} = 0$

43. D. $V(x) = x^3$

$\dfrac{dv}{dx} = 3x^2$

44. E. $\dfrac{ds}{dt} = \dfrac{d}{dt}(2 + 7t - t^2)$

$v(t) = 7 - 2t < 0$

$7 < 2t$

$\dfrac{7}{2} < t$

$4 > \dfrac{7}{2}$

45. C. $v(t) = 7 - 2t = 0$

$7 = 2t$

$t = \dfrac{7}{2}$

46. The growth rate is given by

$b'(t) = 10^4 - 2 \cdot 10^3(t) = 10,000 - 2000t.$

At $t = 0: b'(0) = 10,000$ bacteria/hour

At $t = 5: b'(5) = 0$ bacteria/hour

At $t = 10: b'(10) = -10,000$ bacteria/hour

47. (a) $g'(x) = \dfrac{d}{dx}(x^3) = 3x^2$

$h'(x) = \dfrac{d}{dx}(x^3 - 2) = 3x^2$

$t'(x) = \dfrac{d}{dx}(x^3 + 3) = 3x^2$

(b) The graphs of NDER $g(x)$, NDER $h(x)$, and NDER $t(x)$ are all the same, as shown.
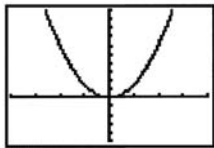
[−4, 4] by [−10, 20]

(c) $f(x)$ must be of the form $f(x) = x^3 + c$, where c is a constant.

(d) Yes. $f(x) = x^3$

(e) Yes. $f(x) = x^3 + 3$

48. For $t > 0$, the speed of the aircraft in meters per second after t seconds is $\dfrac{20}{9}t$. Multiplying by $\dfrac{3600 \text{ sec}}{1 \text{ h}} \cdot \dfrac{1 \text{ km}}{1000 \text{ m}}$, we find that this is equivalent to $8t$ kilometers per hour. Solving $8t = 200$ gives $t = 25$ seconds. The aircraft takes 25 seconds to become airborne, and the distance it travels during this time is $D(25) \approx 694.444$ meters.

49. (a) Assume that f is even. Then,

$f'(-x) = \lim\limits_{h \to 0} \dfrac{f(-x+h) - f(-x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{f(x-h) - f(x)}{h},$

and substituting $k = -h$,

$= \lim\limits_{k \to 0} \dfrac{f(x+k) - f(x)}{-k}$

$= -\lim\limits_{k \to 0} \dfrac{f(x+k) - f(x)}{k} = f'(x)$

So, f' is an odd function.

(b) Assume that f is odd. Then,

$f'(-x) = \lim\limits_{h \to 0} \dfrac{f(-x+h) - f(-x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-f(x-h) + f(x)}{h},$

and substituting $k = -h$,

$= \lim\limits_{k \to 0} \dfrac{-f(x+k) + f(x)}{-k}$

$= \lim\limits_{k \to 0} \dfrac{f(x+k) - f(x)}{k} = f'(x)$

So, f' is an even function.

50. $\dfrac{d}{dx}(fgh) = \dfrac{d}{dx}[f(gh)] = f \cdot \dfrac{d}{dx}(gh) + gh \cdot \dfrac{d}{dx}(f)$

$= f\left(g \cdot \dfrac{dh}{dx} + h \cdot \dfrac{dg}{dx}\right) + gh \cdot \dfrac{df}{dx}$

$= \left(\dfrac{df}{dx}\right)gh + f\left(\dfrac{dg}{dx}\right)h + fg\left(\dfrac{dh}{dx}\right)$

Section 3.5 Derivatives of Trigonometric Functions (pp. 141–147)

Exploration 1   Making a Conjecture with NDER

1. When the graph of sin x is increasing, the graph of NDER (sin x) is positive (above the x-axis).

2. When the graph of sin x is decreasing, the graph of NDER (sin x) is negative (below the x-axis).

3. When the graph of sin x stops increasing and starts decreasing, the graph of NDER (sin x) crosses the x-axis from above to below.

4. The slope of the graph of sin x matches the value of NDER (sin x) at these points.

5. We conjecture that NDER (sin x) = cos x. The graphs coincide, supporting our conjecture.

[−2π, 2π] by [−4, 4]

6. When the graph of cos x is increasing, the graph of NDER (cos x) is positive (above the x-axis).
When the graph of cos x is decreasing, the graph of NDER (cos x) is negative (below the x-axis).
When the graph of cos x stops increasing and starts decreasing, the graph of NDER (cos x) crosses the x-axis from above to below.
The slope of the graph of cos x matches the value of NDER (cos x) at these points.

6. Continued

We conjecture that NDER (cos x) = $-\sin x$. The graphs coincide, supporting our conjecture.
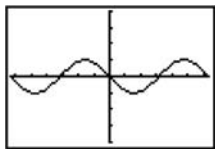
$[-2\pi, 2\pi]$ by $[-4, 4]$

Quick Review 3.5

1. $135° \cdot \dfrac{\pi}{180°} = \dfrac{3\pi}{4} \approx 2.356$

2. $1.7 \cdot \dfrac{180°}{\pi} = \left(\dfrac{306}{\pi}\right)° \approx 97.403°$

3. $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

4. Domain: All reals
Range: $[-1, 1]$

5. Domain: $x \neq \dfrac{k\pi}{2}$ for odd integers k
Range: All reals

6. $\cos a = \pm\sqrt{1 - \sin^2 a} = \pm\sqrt{1 - (-1)^2} = \pm\sqrt{0} = 0$

7. If $\tan a = -1$, then $a = \dfrac{3\pi}{4} + k\pi$ for some integer k,

so $\sin a = \pm\dfrac{1}{\sqrt{2}}$.

8. $\dfrac{1 - \cos h}{h} = \dfrac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} = \dfrac{1 - \cos^2 h}{h(1 + \cos h)}$

$= \dfrac{\sin^2 h}{h(1 + \cos h)}$

9. $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes through (3, 1), so its equation is $y = 12(x - 3) + 1$, or $y = 12x - 35$.

10. $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

Section 3.5 Exercises

1. $\dfrac{d}{dx}(1 + x - \cos x) = 0 + 1 - (-\sin x) = 1 + \sin x$

2. $\dfrac{d}{dx}(2\sin x - \tan x) = 2\cos x - \sec^2 x$

3. $\dfrac{d}{dx}\left(\dfrac{1}{x} + 5\sin x\right) = -\dfrac{1}{x^2} + 5\cos x$

4. $\dfrac{d}{dx}(x\sec x) = x\dfrac{d}{dx}(\sec x) + \sec x\dfrac{d}{dx}(x)$
$= x\sec x\tan x + \sec x$

5. $\dfrac{d}{dx}(4 - x^2\sin x) = \dfrac{d}{dx}(4) - \left[x^2\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(x^2)\right]$

$= 0 - [x^2\cos x + (\sin x)(2x)]$
$= -x^2\cos x - 2x\sin x$

6. $\dfrac{d}{dx}(3x + x\tan x) = \dfrac{d}{dx}(3x) + \left[x\dfrac{d}{dx}(\tan x) + (\tan x)\dfrac{d}{dx}(x)\right]$

$= 3 + x\sec^2 x + \tan x$

7. $\dfrac{d}{dx}\left(\dfrac{4}{\cos x}\right) = \dfrac{d}{dx}(4\sec x) = 4\sec x\tan x$

8. $\dfrac{d}{dx}\dfrac{x}{1 + \cos x} = \dfrac{(1 + \cos x)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}$

$= \dfrac{1 + \cos x + x\sin x}{(1 + \cos x)^2}$

9. $\dfrac{d}{dx}\dfrac{\cot x}{1 + \cot x} = \dfrac{(1 + \cot x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2}$

$= \dfrac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$

$= -\dfrac{\csc^2 x}{(1 + \cot x)^2} = -\dfrac{\csc^2 x\sin^2 x}{(1 + \cot x)^2\sin^2 x} = -\dfrac{1}{(\sin x + \cos x)^2}$

10. $\dfrac{d}{dx}\dfrac{\cos x}{1 + \sin x} = \dfrac{(1 + \sin x)\dfrac{d}{dx}(\cos x) - (\cos x)\dfrac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$

$= \dfrac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$

$= \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$

$= \dfrac{-(1 + \sin x)}{(1 + \sin x)^2}$

$= -\dfrac{1}{1 + \sin x}$

11. $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dx}(5\sin t)$

$v(t) = 5\cos t$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dx}(5\cos t)$

$a(t) = -5\sin t$

The weight starts at 0, goes to 5, and the oscillates between 5 and -5. The period of the motion is 2π. The speed is greatest when $\cos t = \pm 1$ ($t = k\pi$), zero when

$\cos t = 0 \left(t = \dfrac{k\pi}{2}, k \text{ odd}\right)$. The acceleration is greatest

when $\sin t = \pm 1 \left(t = \dfrac{k\pi}{2}, k \text{ odd}\right)$, zero when

$\sin t = 0 (t = k\pi)$.

12. $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dx}(7 \cos t)$

$v(t) = -7\sin t$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dx}(-7 \sin t)$

$a(t) = -7\cos t$

The weight starts at 7, goes to −7, and then oscillates between −7 and 7. The period of the motion is 2π. The speed is greatest when $\sin t = \pm 1$ $\left(t = \dfrac{k\pi}{2}, \; k \text{ odd}\right)$, zero when $\sin t = 0$ $(t = k\pi)$. The acceleration is greatest when $\cos t = \pm 1 (t = k\pi)$, zero when $\cos t = 0\left(t = \dfrac{k\pi}{2}, \; k \text{ odd}\right)$.

13. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(2 + 3\sin t)$

$v(t) = 3\cos t, \; \text{speed} = |3\cos t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(3\cos t) = -3\sin t$

(b) $v\left(\dfrac{\pi}{4}\right) = 3\cos\left(\dfrac{\pi}{4}\right) = \dfrac{3\sqrt{2}}{2}, \; \text{speed} = \dfrac{3\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right) = -3\sin\left(\dfrac{\pi}{4}\right) = -\dfrac{3\sqrt{2}}{2}$

(c) The body starts at 2, goes up to 5, goes down to −1, and then oscillates between −1 and 5. The period of motion is 2π.

14. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(1 - 4\cos t)$

$v(t) = 4 \sin t, \; \text{speed} = |4 \sin t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(4 \sin t)$

$a(t) = 4 \cos t$

(b) $v\left(\dfrac{\pi}{4}\right) = 4\sin\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}, \; \text{speed} = 2\sqrt{2}$

$a\left(\dfrac{\pi}{4}\right) = 4\cos t\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

(c) The body starts at −3, goes up to 5, and then oscillates between 5 and −3. The period of the motion is 2π.

15. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(2\sin t + 3\cos t)$

$v(t) = 2 \cos t - 3 \sin t, \; \text{speed} = |2 \cos t - 3 \sin t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(2\cos t - 3\sin t)$

$a(t) = -2 \sin t - 3 \cos t$

(b) $v\left(\dfrac{\pi}{4}\right) = 2\cos\dfrac{\pi}{4} - 3\sin\dfrac{\pi}{4}$

$v\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

$\text{speed} = \dfrac{\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right) = -2\sin\dfrac{\pi}{4} - 3\cos\dfrac{\pi}{4}$

$a\left(\dfrac{\pi}{4}\right) = \dfrac{-5\sqrt{2}}{2}$

(c) The body starts at 3, goes to $3.606(\sqrt{13})$, and then oscillates between −3.606 and 3.606. The period of the motion is 2π.

16. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(\cos t - 3\sin t)$

$v(t) = -\sin t - 3\cos t, \; \text{speed} = |-\sin t - 3\cos t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(-\sin t - 3\cos t)$

$a(t) = -\cos t + 3\sin t$

(b) $v\left(\dfrac{\pi}{4}\right) = -\sin\dfrac{\pi}{4} - 3\cos\dfrac{\pi}{4} = -2\sqrt{2}$

$\text{speed} = 2\sqrt{2}$

$a\left(\dfrac{\pi}{4}\right) = -\sin\dfrac{\pi}{4} - 3\cos\dfrac{\pi}{4} = \sqrt{2}$

(c) The body starts at 1, goes to $-3.162(-\sqrt{10})$, and then oscillates between 3.162 and −3.162. The period of the motion is 2π.

17. $j(t) = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

$f(t) = 2 \cos t$

$f'(t) = -2 \sin t$

$f''(t) = -2 \cos t$

$f'''(t) = 2 \sin t$

18. $j(t) = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

$f(t) = 1 + 2 \cos t$

$f'(t) = -2 \sin t$

$f''(t) = -2 \cos t$

$f'''(t) = 2 \sin t$

19. $j(t) = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

$f(t) = \sin t - \cos t$

$f'(t) = \cos t + \sin t$

$f''(t) = -\sin t + \cos t$

$f'''(t) = -\cos t - \sin t$

20. $j(t) = \dfrac{da}{dt} = \dfrac{d^3 s}{dt}$

$f(t) = 2 + 2\sin t$

$f'(t) = 2\cos t$

$f''(t) = -2\sin t$

$f'''(t) = -2\cos t$

21. $y = \sin x + 3$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin x + 3) = \cos x$

$y(\pi) = \sin \pi + 3 = 3$

$y'(\pi) = \cos \pi = -1$

tangent: $y = -1(x - \pi) + 3 = -x + \pi + 3$

normal: $m_2 = -\dfrac{1}{m_1} = 1$

$y = (x - \pi) + 3$

22. $y = \sec x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sec x) = \sec x \tan x$

$y\left(\dfrac{\pi}{4}\right) = \sec \dfrac{\pi}{4} = 1.414$

$y'\left(\dfrac{\pi}{4}\right) = \sec \dfrac{\pi}{4} \tan \dfrac{\pi}{4} = 1.414$

tangent: $y = 1.414(x - \dfrac{\pi}{4}) + 1.414$

$y = 1.414x + 0.303$

normal: $m_2 = -\dfrac{1}{m_1} = -0.707$

$y = -0.707(x - \dfrac{\pi}{4}) + 1.414 = -0.707x + 1.970$

23. $y = x^2 \sin x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x$

$y(3) = (3)^2 \sin 3 = 1.270$

$y'(3) = 2(3)\sin 3 + (3)^2 \cos 3 = -8.063$

tangent: $y = -8.063(x - 3) + 1.270 = -8.063x + 25.460$

normal: $m_2 = -\dfrac{1}{m_1} = 0.124$

$y = 0.124(x - 3) + 1.270$

$y = 0.124x + 0.898$

24. $\displaystyle\lim_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$

$= \displaystyle\lim_{h \to 0} \dfrac{\cos x (\cos h - 1) - \sin x \sin h}{h}$

$= \displaystyle\lim_{h \to 0} \left((\cos x)\dfrac{\cos h - 1}{h} - (\sin x)\dfrac{\sin h}{h} \right)$

$= (\cos x)\left(\displaystyle\lim_{h \to 0} \dfrac{\cos h - 1}{h} \right) - (\sin x)\left(\displaystyle\lim_{h \to 0} \dfrac{\sin h}{h} \right)$

$= (\cos x)(0) - (\sin x)(1) = -\sin x$

25. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} = \dfrac{(\cos x)\dfrac{d}{dx}(\sin x) - (\sin x)\dfrac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} = \dfrac{(\cos x)\dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \dfrac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$

$= \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

26. (a) $\dfrac{d}{dx}\cot x = \dfrac{d}{dx}\dfrac{\cos x}{\sin x}$

$= \dfrac{(\sin x)\dfrac{d}{dx}(\cos x) - (\cos x)\dfrac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\dfrac{1}{\sin^2 x} = -\csc^2 x$

(b) $\dfrac{d}{dx}\csc x = \dfrac{d}{dx}\dfrac{1}{\sin x}$

$= \dfrac{(\sin x)\dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \dfrac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x}$

$= -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

27. $\dfrac{d}{dx}\sec x = \sec x \tan x$ which is 0 at $x = 0$, so the slope of the

tangent line is 0. $\dfrac{d}{dx}\cos x = -\sin x$ which is 0 at $x = 0$,

so the slope of the tangent line is 0.

28. $\dfrac{d}{dx}\tan x = \sec^2 x = \dfrac{1}{\cos^2 x}$, which is never 0.

$\dfrac{d}{dx}\cot x = -\csc^2 x = -\dfrac{1}{\sin^2 x}$, which is never 0.

29. $y'(x) = \dfrac{d}{dx}(\sqrt{2}\cos x) = -\sqrt{2}\sin x$

$y'\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\sin\dfrac{\pi}{4} = -\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = -1$

The tangent line has slope -1 and passes

through $\left(\dfrac{\pi}{4},\ \sqrt{2}\cos\dfrac{\pi}{4}\right) = \left(\dfrac{\pi}{4},\ 1\right)$, so its equation is

$y = -1\left(x - \dfrac{\pi}{4}\right) + 1$, or $y = -x + \dfrac{\pi}{4} + 1$.

The normal line has slope 1 and passes through $\left(\dfrac{\pi}{4},\ 1\right)$,

so its equation is $y = 1\left(x - \dfrac{\pi}{4}\right) + 1$, or $y = x + 1 - \dfrac{\pi}{4}$.

30. $y'(x) = \dfrac{d}{dx}\tan x = \sec^2 x$

$y'(x) = \dfrac{d}{dx}(2x) = 2$

$\sec^2 x = 2$

$\sec x = \pm\sqrt{2}$

$\cos x = \pm\dfrac{1}{\sqrt{2}}$

On $\left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$, the solutions are $x = \pm\dfrac{\pi}{4}$. The points on the

curve are $\left(-\dfrac{\pi}{4},\ -1\right)$ and $\left(\dfrac{\pi}{4},\ 1\right)$.

31. $y'(x) = \dfrac{d}{dx}(4 + \cot x - 2\csc x)$

$= 0 - \csc^2 x + 2\csc x\cot x$

$= -\csc^2 x + 2\csc x\cot x$

(a) $y'\left(\dfrac{\pi}{2}\right) = -\csc^2\dfrac{\pi}{2} + 2\csc\dfrac{\pi}{2}\cot\dfrac{\pi}{2}$

$= -1^2 + 2(1)(0) = -1$

The tangent line has slope -1 and passes through

$P\left(\dfrac{\pi}{2},\ 2\right)$. Its equation is $y = -1\left(x - \dfrac{\pi}{2}\right) + 2$, or

$y = -x + \dfrac{\pi}{2} + 2$.

(b) $f'(x) = 0$

$-\csc^2 x + 2\csc x\cot x = 0$

$-\dfrac{1}{\sin^2 x} + \dfrac{2\cos x}{\sin^2 x} = 0$

$\dfrac{1}{\sin^2 x}(2\cos x - 1) = 0$

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}$ at point Q

$y\left(\dfrac{\pi}{3}\right) = 4 + \cot\dfrac{\pi}{3} - 2\csc\dfrac{\pi}{3}$

$= 4 + \dfrac{1}{\sqrt{3}} - 2\left(\dfrac{2}{\sqrt{3}}\right)$

$= 4 - \dfrac{3}{\sqrt{3}} = 4 - \sqrt{3}$

The coordinates of Q are $\left(\dfrac{\pi}{3},\ 4 - \sqrt{3}\right)$.

The equation of the horizontal line is $y = 4 - \sqrt{3}$.

32. $y'(x) = \dfrac{d}{dx}(1 + \sqrt{2}\csc x + \cot x)$

$= 0 + \sqrt{2}(-\csc x\cot x) + (-\csc^2 x)$

$= -\sqrt{2}\csc x\cot x - \csc^2 x$

(a) $y'\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\csc\dfrac{\pi}{4}\cot\dfrac{\pi}{4} - \csc^2\dfrac{\pi}{4}$

$= -\sqrt{2}(\sqrt{2})(1) - (\sqrt{2})^2$

$= -2 - 2 = -4$

The tangent line has slope -4 and passes through

$P\left(\dfrac{\pi}{4},\ 4\right)$. Its equation is $y = -4\left(x - \dfrac{\pi}{4}\right) + 4$, or

$y = -4x + \pi + 4$.

(b) $y'(x) = 0$

$-\sqrt{2}\csc x\cot x - \csc^2 x = 0$

$-\dfrac{\sqrt{2}\cos x}{\sin^2 x} - \dfrac{1}{\sin^2 x} = 0$

$-\dfrac{1}{\sin^2 x}(\sqrt{2}\cos x + 1) = 0$

$\cos x = -\dfrac{1}{\sqrt{2}}$

$x = \dfrac{3\pi}{4}$ at point Q

$y\left(\dfrac{3\pi}{4}\right) = 1 + \sqrt{2}\csc\dfrac{3\pi}{4} + \cot\dfrac{3\pi}{4}$

$= 1 + \sqrt{2}(\sqrt{2}) + (-1)$

$= 2$

The coordinates of Q are $\left(\dfrac{3\pi}{4},\ 2\right)$.

The equation of the horizontal line is $y = 2$.

33. (a) Velocity: $s'(t) = -2\cos t$ m/sec

Speed: $|s'(t)| = |2\cos t|$ m/sec

Acceleration: $s''(t) = 2\sin t$ m/sec^2

Jerk: $s'''(t) = 2\cos t$ m/sec^3

(b) Velocity: $-2\cos\dfrac{\pi}{4} = -\sqrt{2}$ m/sec

Speed: $\left|-\sqrt{2}\right| = \sqrt{2}$ m/sec

Acceleration: $2\sin\dfrac{\pi}{4} = \sqrt{2}$ m/sec^2

Jerk: $2\cos\dfrac{\pi}{4} = \sqrt{2}$ m/sec^3

(c) The body starts at 2, goes to 0 and then oscillates between 0 and 4.

Speed:

Greatest when $\cos t = \pm1$ (or $t = k\pi$), at the center of the interval of motion.

Zero when $\cos t = 0$ (or $t = \dfrac{k\pi}{2}$, k odd), at the endpoints of the interval of motion.

Acceleration:

Greatest (in magnitude) when $\sin t = \pm1$

(or $t = \dfrac{k\pi}{2}$, k odd)

Zero when $\sin t = 0$ (or $t = k\pi$)

Jerk:

Greatest (in magnitude) when $\cos t = \pm1$ (or $t = k\pi$).

Zero when $\cos t = 0$ (or $t = \dfrac{k\pi}{2}$, k odd)

34. (a) Velocity: $s'(t) = \cos t - \sin t$ m/sec

Speed: $s'(t) = |\cos t - \sin t|$ m/sec

Acceleration: $s''(t) = -\sin t - \cos t$ m/sec^2

Jerk: $s'''(t) = -\cos t - \sin t$ m/sec^3

(b) Velocity: $\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{4} = 0$ m/sec

Speed: $|0| = 0$ m/sec

Acceleration: $-\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4} = -\sqrt{2}$ m/sec^2

Jerk: $-\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{4} = 0$ m/sec^3

(c) The body starts at 1, goes to $\sqrt{2}$ and then oscillates between $\pm\sqrt{2}$.

Speed:

Greatest when $t = \dfrac{3\pi}{4} + k\pi$

Zero when $t = \dfrac{\pi}{4} + k\pi$

Acceleration:

Greatest (in magnitude) when $t = \dfrac{\pi}{4} + k\pi$

Zero when $t = \dfrac{3\pi}{4} + k\pi$

Jerk:

Greatest (in magnitude) when $t = \dfrac{3\pi}{4} + k\pi$

Zero when $t = \dfrac{\pi}{4} + k\pi$

35. $y' = \dfrac{d}{dx}\csc x = -\csc x \cot x$

$y'' = \dfrac{d}{dx}(-\csc x \cot x)$

$\quad = -(\csc x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(\csc x)$

$\quad = -(\csc x)(-\csc^2 x) - (\cot x)(-\csc x \cot x)$

$\quad = \csc^3 x + \csc x \cot^2 x$

36. $y' = \dfrac{d}{d\theta}(\theta \tan\theta)$

$\quad = \theta \dfrac{d}{d\theta}(\tan\theta) + (\tan\theta)\dfrac{d}{d\theta}(\theta)$

$\quad = \theta \sec^2\theta + \tan\theta$

$y'' = \dfrac{d}{d\theta}(\theta \sec^2\theta + \tan\theta)$

$\quad = \theta \dfrac{d}{d\theta}[(\sec\theta)(\sec\theta)] + (\sec^2\theta)\dfrac{d}{d\theta}(\theta) + \dfrac{d}{d\theta}(\tan\theta)$

$\quad = \theta\left[(\sec\theta)\dfrac{d}{d\theta}(\sec\theta) + (\sec\theta)\dfrac{d}{d\theta}(\sec\theta)\right] + \sec^2\theta + \sec^2\theta$

$\quad = 2\theta \sec^2\theta \tan\theta + 2\sec^2\theta$

$\quad = (2\theta \tan\theta + 2)(\sec^2\theta)$

or, writing in terms of sines and cosines,

$\quad = \dfrac{2 + 2\theta \tan\theta}{\cos^2\theta}$

$\quad = \dfrac{2\cos\theta + 2\theta \sin\theta}{\cos^3\theta}$

37. Continuous:

Note that $g(0) = \lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \cos x = \cos(0) = 1$, and $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (x + b) = b$. We require $\lim\limits_{x \to 0^-} g(x) = g(0)$, so $b = 1$. The function is continuous if $b = 1$.

Differentiable:
For $b = 1$, the left-hand derivative is 1 and the right-hand derivative is $-\sin(0) = 0$, so the function is not differentiable. For other values of b, the function is discontinuous at $x = 0$ and there is no left-hand derivative. So, there is no value of b that will make the function differentiable at $x = 0$.

38. Observe the pattern:

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d^2}{dx^2}\cos x = -\cos x$$

$$\frac{d^3}{dx^3}\cos x = \sin x$$

$$\frac{d^4}{dx^4}\cos x = \cos x$$

$$\frac{d^5}{dx^5}\cos x = -\sin x$$

$$\frac{d^6}{dx^6}\cos x = -\cos x$$

$$\frac{d^7}{dx^7}\cos x = \sin x$$

$$\frac{d^8}{dx^8}\cos x = \cos x$$

Continuing the pattern, we see that

$$\frac{d^n}{dx^n}\cos x = \sin x \text{ when } n = 4k + 3 \text{ for any whole}$$

number k.

Since $999 = 4(249) + 3$, $\dfrac{d^{999}}{dx^{999}}\cos x = \sin x$.

39. Observe the pattern:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d^2}{dx^2}\sin x = -\sin x$$

$$\frac{d^3}{dx^3}\sin x = -\cos x$$

$$\frac{d^4}{dx^4}\sin x = \sin x$$

$$\frac{d^5}{dx^5}\sin x = \cos x$$

$$\frac{d^6}{dx^6}\sin x = -\sin x$$

$$\frac{d^7}{dx^7}\sin x = -\cos x$$

$$\frac{d^8}{dx^8}\sin x = \sin x$$

Continuing the pattern, we see that

$$\frac{d^n}{dx^n}\sin x = \cos x \text{ when } n = 4k + 1 \text{ for any whole number } k.$$

Since $725 = 4(181) + 1$, $\dfrac{d^{725}}{dx^{725}}\sin x = \cos x$.

40. The line is tangent to the graph of $y = \sin x$ at (0, 0). Since $y'(0) = \cos(0) = 1$, the line has slope 1 and its equation is $y = x$.

41. (a) Using $y = x$, $\sin(0.12) \approx 0.12$.

   (b) $\sin(0.12) \approx 0.1197122$; The approximation is within 0.0003 of the actual value.

42. $\dfrac{d}{dx}\sin 2x = \dfrac{d}{dx}(2\sin x \cos x)$

$$= 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\sin x)\right]$$

$$= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]$$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2\cos 2x$$

43. $\dfrac{d}{dx}\cos 2x = \dfrac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]$

$$= \left[(\cos x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\cos x)\right] -$$

$$\left[(\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)\right]$$

$$= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)$$

$$= -4\sin x \cos x$$

$$= -2(2\sin x \cos x)$$

$$= -2\sin 2x$$

44. True. The derivative is positive at $t = \dfrac{3\pi}{4}$.

45. False. The velocity is negative and the speed is positive

at $t = \dfrac{\pi}{4}$.

46. A. $y = \sin x + \cos x$

$$\frac{dy}{dx_1} = \frac{d}{dx} = \cos x - \sin x$$

$$y(\pi) = \sin \pi + \cos \pi = -1$$

$$y'(\pi) = \cos \pi - \sin \pi = -1$$

$$y = -1(x - \pi) - 1$$

$$y = -x + \pi - 1$$

47. B. See 46.

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$

$$y = (x - \pi) - 1$$

48. C. $y = x \sin x$

$$y' = \sin x + x \cos x$$

$$y'' = \cos x + \cos x - x \sin x$$

$$= -x \sin x + 2\cos x$$

49. C. $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(3 + \sin t)$

$$v(t) = \cos t = 0$$

$$t = \frac{\pi}{2}$$

50. (a)
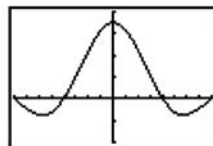
[−360, 360] by [−0.01, 0.02]

The limit is $\dfrac{\pi}{180}$ because this is the conversion factor for changing from degrees to radians.

50. Continued

(b)
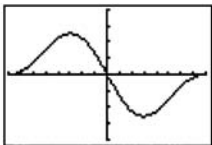
[−360, 360] by [−0.02, 0.02]

This limit is still 0.

(c) $\dfrac{d}{dx} \sin x = \lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sin x(\cos h - 1) + \cos x \sin h}{h}$

$= \left(\lim\limits_{h \to 0} \sin x \right)\left(\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} \right) + \left(\lim\limits_{h \to 0} \cos x \right)\left(\lim\limits_{h \to 0} \dfrac{\sin h}{h} \right)$

$= (\sin x)(0) + (\cos x)\left(\dfrac{\pi}{180} \right)$

$= \dfrac{\pi}{180} \cos x$

(d) $\dfrac{d}{dx} \cos x = \lim\limits_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \lim\limits_{h \to 0} \dfrac{(\cos x)(\cos h - 1) - \sin x \sin h}{h}$

$= \left(\lim\limits_{h \to 0} \cos x \right)\left(\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} \right) - \left(\lim\limits_{h \to 0} \sin x \right)\left(\lim\limits_{h \to 0} \dfrac{\sin h}{h} \right)$

$= (\cos x)(0) - (\sin x)\left(\dfrac{\pi}{180} \right)$

$= -\dfrac{\pi}{180} \sin x$

(e) $\dfrac{d^2}{dx^2} \sin x = \dfrac{d}{dx} \dfrac{\pi}{180} \cos x = \dfrac{\pi}{180}\left(-\dfrac{\pi}{180} \sin x \right)$

$= -\dfrac{\pi^2}{180^2} \sin x$

$\dfrac{d^3}{dx^3} \sin x = \dfrac{d}{dx}\left(-\dfrac{\pi^2}{180^2} \sin x \right) = -\dfrac{\pi^2}{180^2}\left(\dfrac{\pi}{180} \cos x \right)$

$= -\dfrac{\pi^3}{180^3} \cos x$

$\dfrac{d^2}{dx^2} \cos x = \dfrac{d}{dx}\left(-\dfrac{\pi}{180} \sin x \right) = -\dfrac{\pi}{180}\left(\dfrac{\pi}{180} \cos x \right)$

$= -\dfrac{\pi^2}{180^2} \cos x$

$\dfrac{d^3}{dx^3} \cos x = \dfrac{d}{dx}\left(-\dfrac{\pi^2}{180^2} \cos x \right) = -\dfrac{\pi^2}{180^2}\left(-\dfrac{\pi}{180} \sin x \right)$

$= \dfrac{\pi^3}{180^3} \sin x$

51. $\lim\limits_{h \to 0} \dfrac{(\cos h - 1)}{h} = \lim\limits_{h \to 0} \dfrac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$

$= \lim\limits_{h \to 0} \dfrac{\cos^2 h - 1}{h(\cos h + 1)}$

$= \lim\limits_{h \to 0} \dfrac{-\sin^2 h}{h(\cos h + 1)}$

$= -\left(\lim\limits_{h \to 0} \dfrac{\sin h}{h} \right)\left(\lim\limits_{h \to 0} \dfrac{\sin h}{\cos h + 1} \right)$

$= -(1)\left(\dfrac{0}{2} \right) = 0$

52. $y' = \dfrac{d}{dx}(A \sin x + B \cos x) = A \cos x - B \sin x$

$y'' = \dfrac{d}{dx}(A \cos x - B \sin x) = -A \sin x - B \cos x$

Solve: $\qquad\qquad\qquad y'' - y = \sin x$

$\qquad (-A \sin x - B \cos x) - (A \sin x + B \cos x) = \sin x$

$\qquad\qquad\qquad\qquad\qquad -2A \sin x - 2B \cos x = \sin x$

At $x = \dfrac{\pi}{2}$, this gives $-2A = 1$, so $A = -\dfrac{1}{2}$.

At $x = 0$, we have $-2B = 0$, so $B = 0$.

Thus, $A = -\dfrac{1}{2}$ and $B = 0$.

Section 3.6 Chain Rule (pp. 148–156)

Quick Review 3.6

1. $f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1)$

2. $f(g(h(x))) = f(g(7x)) = f((7x)^2 + 1)$
$= \sin[(7x)^2 + 1] = \sin(49x^2 + 1)$

3. $(g \circ h)(x) = g(h(x)) = g(7x) = (7x)^2 + 1 = 49x^2 + 1$

4. $(h \circ g)(x) = h(g(x)) = h(x^2 + 1) = 7(x^2 + 1) = 7x^2 + 7$

5. $f\left(\dfrac{g(x)}{h(x)} \right) = f\left(\dfrac{x^2 + 1}{7x} \right) = \sin \dfrac{x^2 + 1}{7x}$

6. $\sqrt{\cos x + 2} = g(\cos x) = g(f(x))$

7. $\sqrt{3\cos^2 x + 2} = g(3\cos^2 x) = g(h(\cos x)) = g(h(f(x)))$

8. $3\cos x + 6 = 3(\cos x + 2) = 3(\sqrt{\cos x + 2})^2$
$= h(\sqrt{\cos x + 2}) = h(g(\cos x)) = h(g(f(x)))$

9. $\cos 27x^4 = f(27x^4) = f(3(3x^2)^2) = f(h(3x^2)) = f(h(h(x)))$

10. $\cos \sqrt{2 + 3x^2} = \cos \sqrt{3x^2 + 2} = f(\sqrt{3x^2 + 2})$
$= f(g(3x^2)) = f(g(h(x)))$

Section 3.6 Exercises

1. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \sin u \qquad\qquad u = 3x + 1$

$\quad \dfrac{dy}{du} = \cos u \qquad \dfrac{du}{dx} = 3$

$\quad \dfrac{dy}{dx} = 3\cos(3x+1)$

2. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \sin u \qquad\qquad u = 7 - 5x$

$\quad \dfrac{dy}{du} = \cos u \qquad \dfrac{du}{dx} = -5$

$\quad \dfrac{dy}{dx} = -5\cos(7 - 5x)$

3. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \cos u \qquad\qquad u = \sqrt{3}x$

$\quad \dfrac{dy}{du} = -\sin u \qquad \dfrac{du}{dx} = \sqrt{3}$

$\quad \dfrac{dy}{dx} = -\sqrt{3}\sin(\sqrt{3}x)$

4. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \tan u \qquad\qquad u = 2x - x^3$

$\quad \dfrac{dy}{du} = \sec^2 u \qquad \dfrac{du}{dx} = 2 - 3x^2$

$\quad \dfrac{dy}{dx} = (2 - 3x^2)\sec^2(2x - x^3)$

5. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = u^2 \qquad\qquad u = \dfrac{\sin x}{1 + \cos x}$

$\quad \dfrac{dy}{du} = 2u \qquad \dfrac{du}{dx} = \dfrac{\sin x}{(1 + \cos x)^2}$

$\quad \dfrac{dy}{dx} = \dfrac{2\sin x}{(1 + \cos x)^2}$

6. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = 5\cot u \qquad\qquad u = \dfrac{2}{x}$

$\quad \dfrac{dy}{du} = -5\csc^2 u \qquad \dfrac{du}{dx} = -\dfrac{2}{x^2}$

$\quad \dfrac{dy}{dx} = \dfrac{10}{x^2}\csc^2\left(\dfrac{2}{x}\right)$

7. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \cos u \qquad\qquad u = \sin x$

$\quad \dfrac{dy}{du} = -\sin u \qquad \dfrac{du}{dx} = \cos x$

$\quad \dfrac{dy}{dx} = -\sin(\sin x)\cos x$

8. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad y = \sec u$

$\quad \dfrac{dy}{du} = \sec u \tan u$

$\quad u = \tan x$

$\quad \dfrac{du}{dx} = \sec^2 x$

$\quad \dfrac{dy}{dx} = \sec(\tan x)\tan(\tan x)\sec^2 x$

9. $\dfrac{ds}{dt} = \dfrac{d}{dt}\cos\left(\dfrac{\pi}{2} - 3t\right)$

$\quad = \left[-\sin\left(\dfrac{\pi}{2} - 3t\right)\right]\dfrac{d}{dt}\left(\dfrac{\pi}{2} - 3t\right)$

$\quad = \left[-\sin\left(\dfrac{\pi}{2} - 3t\right)\right](-3)$

$\quad = 3\sin\left(\dfrac{\pi}{2} - 3t\right)$

10. $\dfrac{ds}{dt} = \dfrac{d}{dt}[t\cos(\pi - 4t)]$

$\quad = (t)\dfrac{d}{dt}[\cos(\pi - 4t)] + \cos(\pi - 4t)\dfrac{d}{dt}(t)$

$\quad = t[-\sin(\pi - 4t)]\dfrac{d}{dt}(\pi - 4t) + \cos(\pi - 4t)(1)$

$\quad = t[-\sin(\pi - 4t)](-4) + \cos(\pi - 4t)$

$\quad = 4t\sin(\pi - 4t) + \cos(\pi - 4t)$

11. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t\right)$

$\quad = \dfrac{4}{3\pi}(\cos 3t)\dfrac{d}{dt}(3t) + \dfrac{4}{5\pi}(-\sin 5t)\dfrac{d}{dt}(5t)$

$\quad = \dfrac{4}{3\pi}(\cos 3t)(3) + \dfrac{4}{5\pi}(-\sin 5t)(5)$

$\quad = \dfrac{4}{\pi}\cos 3t - \dfrac{4}{\pi}\sin 5t$

12. $\dfrac{ds}{dt} = \dfrac{d}{dt}\left[\sin\left(\dfrac{3\pi}{2}t\right) + \cos\left(\dfrac{7\pi}{4}t\right)\right]$

$\quad = \cos\left(\dfrac{3\pi}{2}t\right)\dfrac{d}{dt}\left(\dfrac{3\pi}{2}t\right) - \sin\left(\dfrac{7\pi}{4}t\right)\dfrac{d}{dt}\left(\dfrac{7\pi}{4}t\right)$

$\quad = \dfrac{3\pi}{2}\cos\left(\dfrac{3\pi}{2}t\right) - \dfrac{7\pi}{4}\sin\left(\dfrac{7\pi}{4}t\right)$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x + \sqrt{x})^{-2} = -2(x + \sqrt{x})^{-3}\dfrac{d}{dx}(x + \sqrt{x})$

$= -2(x + \sqrt{x})^{-3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\csc x + \cot x)^{-1}$

$= -(\csc x + \cot x)^{-2}\dfrac{d}{dx}(\csc x + \cot x)$

$= -\dfrac{1}{(\csc x + \cot x)^2}(-\cot x\csc x - \csc^2 x)$

$= \dfrac{(\csc x)(\cot x + \csc x)}{(\csc x + \cot x)^2} = \dfrac{\csc x}{\csc x + \cot x}$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^{-5} x - \cos^3 x)$

$= (-5\sin^{-6} x)\dfrac{d}{dx}(\sin x) - (3\cos^2 x)\dfrac{d}{dx}(\cos x)$

$= -5\sin^{-6} x\cos x + 3\cos^2 x\sin x$

16. $\dfrac{dy}{dx} = \dfrac{d}{dx}[x^3(2x - 5)^4]$

$= (x^3)\dfrac{d}{dx}(2x - 5)^4 + (2x - 5)^4\dfrac{d}{dx}(x^3)$

$= (x^3)(4)(2x - 5)^3\dfrac{d}{dx}(2x - 5) + (2x - 5)^4(3x^2)$

$= (x^3)(4)(2x - 5)^3(2) + 3x^2(2x - 5)^4$

$= 8x^3(2x - 5)^3 + 3x^2(2x - 5)^4$

$= x^2(2x - 5)^3[8x + 3(2x - 5)]$

$= x^2(2x - 5)^3(14x - 15)$

17. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^3 x\tan 4x)$

$= (\sin^3 x)\dfrac{d}{dx}(\tan 4x) + (\tan 4x)\dfrac{d}{dx}(\sin^3 x)$

$= (\sin^3 x)(\sec^2 4x)\dfrac{d}{dx}(4x) + (\tan 4x)(3\sin^2 x)\dfrac{d}{dx}(\sin x)$

$= (\sin^3 x)(\sec^2 4x)(4) + (\tan 4x)(3\sin^2 x)(\cos x)$

$= 4\sin^3 x\sec^2 4x + 3\sin^2 x\cos x\tan 4x$

18. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4\sqrt{\sec x + \tan x})$

$= 4 \cdot \dfrac{1}{2\sqrt{\sec x + \tan x}}\dfrac{d}{dx}(\sec x + \tan x)$

$= \dfrac{2}{\sqrt{\sec x + \tan x}}(\sec x\tan x + \sec^2 x)$

$= 2\sec x\dfrac{\sec x + \tan x}{\sqrt{\sec x + \tan x}}$

$= 2\sec x\sqrt{\sec x + \tan x}$

19. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{3}{\sqrt{2x + 1}}\right)$

$= \dfrac{(\sqrt{2x + 1})\dfrac{d}{dx}(3) - 3\dfrac{d}{dx}(\sqrt{2x + 1})}{(\sqrt{2x + 1})^2}$

$= \dfrac{(\sqrt{2x + 1})(0) - 3\left(\dfrac{1}{2\sqrt{2x + 1}}\right)\dfrac{d}{dx}(2x + 1)}{2x + 1}$

$= \dfrac{-3\left(\dfrac{1}{2\sqrt{2x + 1}}\right)(2)}{2x + 1}$

$= -\dfrac{3}{(2x + 1)\sqrt{2x + 1}}$

$= -3(2x + 1)^{-3/2}$

20. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{\sqrt{1 + x^2}}$

$= \dfrac{(\sqrt{1 + x^2})\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(\sqrt{1 + x^2})}{(\sqrt{1 + x^2})^2}$

$= \dfrac{(\sqrt{1 + x^2})(1) - x\left(\dfrac{1}{2\sqrt{1 + x^2}}\right)\dfrac{d}{dx}(1 + x^2)}{1 + x^2}$

$= \dfrac{\sqrt{1 + x^2} - x\left(\dfrac{1}{2\sqrt{1 + x^2}}\right)(2x)}{1 + x^2}$

$= \dfrac{(1 + x^2) - x^2}{(1 + x^2)(\sqrt{1 + x^2})}$

$= (1 + x^2)^{-3/2}$

21. The last step here uses the identity $2\sin a\cos a = \sin 2a$.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\sin^2(3x - 2)$

$= 2\sin(3x - 2)\dfrac{d}{dx}\sin(3x - 2)$

$= 2\sin(3x - 2)\cos(3x - 2)\dfrac{d}{dx}(3x - 2)$

$= 2\sin(3x - 2)\cos(3x - 2)(3)$

$= 6\sin(3x - 2)\cos(3x - 2)$

$= 3\sin(6x - 4)$

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1 + \cos 2x)^2 = 2(1 + \cos 2x)\dfrac{d}{dx}(1 + \cos 2x)$

$= 2(1 + \cos 2x)(-\sin 2x)\dfrac{d}{dx}(2x)$

$= 2(1 + \cos 2x)(-\sin 2x)(2)$

$= -4(1 + \cos 2x)(\sin 2x)$

23. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1+\cos^2 7x)^3$

$= 3(1+\cos^2 7x)^2 \dfrac{d}{dx}(1+\cos^2 7x)$

$= 3(1+\cos^2 7x)^2 (2\cos 7x)\dfrac{d}{dx}(\cos 7x)$

$= 3(1+\cos^2 7x)^2 (2\cos 7x)(-\sin 7x)\dfrac{d}{dx}(7x)$

$= 3(1+\cos^2 7x)^2 (2\cos 7x)(-\sin 7x)(7)$

$= -42(1+\cos^2 7x)^2 \cos 7x \sin 7x$

24. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sqrt{\tan 5x}) = \dfrac{1}{2\sqrt{\tan 5x}}\dfrac{d}{dx}\tan 5x$

$= \dfrac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)\dfrac{d}{dx}(5x)$

$= \dfrac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)(5)$

$= \dfrac{5\sec^2 5x}{2\sqrt{\tan 5x}}$ or $\dfrac{5}{2}(\tan 5x)^{-1/2}\sec^2 5x$

25. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\tan(2-\theta) = \sec^2(2-\theta)\dfrac{d}{d\theta}(2-\theta)$

$= \sec^2(2-\theta)(-1) = -\sec^2(2-\theta)$

26. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(\sec 2\theta \ \tan 2\theta)$

$= (\sec 2\theta)\dfrac{d}{d\theta}(\tan 2\theta) + (\tan 2\theta)\dfrac{d}{d\theta}(\sec 2\theta)$

$= (\sec 2\theta)(\sec^2 2\theta)\dfrac{d}{d\theta}(2\theta) + (\tan 2\theta)(\sec 2\theta \tan 2\theta)\dfrac{d}{d\theta}(2\theta)$

$= 2\sec^3 2\theta + 2\sec 2\theta \tan^2 2\theta$

27. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\sqrt{\theta\sin\theta} = \dfrac{1}{2\sqrt{\theta\sin\theta}}\dfrac{d}{d\theta}(\theta\sin\theta)$

$= \dfrac{1}{2\sqrt{\theta\sin\theta}}\left[\theta\dfrac{d}{d\theta}(\sin\theta) + (\sin\theta)\dfrac{d}{d\theta}(\theta)\right]$

$= \dfrac{1}{2\sqrt{\theta\sin\theta}}(\theta\cos\theta + \sin\theta)$

$= \dfrac{\theta\cos\theta + \sin\theta}{2\sqrt{\theta\sin\theta}}$

28. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(2\theta\sqrt{\sec\theta})$

$= (2\theta)\dfrac{d}{d\theta}(\sqrt{\sec\theta}) + (\sqrt{\sec\theta})\dfrac{d}{d\theta}(2\theta)$

$= (2\theta)\left(\dfrac{1}{2\sqrt{\sec\theta}}\right)\dfrac{d}{d\theta}(\sec\theta) + 2\sqrt{\sec\theta}$

$= (2\theta)\left(\dfrac{1}{2\sqrt{\sec\theta}}\right)(\sec\theta\tan\theta) + 2\sqrt{\sec\theta}$

$= \theta(\sqrt{\sec\theta})(\tan\theta) + 2\sqrt{\sec\theta}$

$= \sqrt{\sec\theta}(\theta\tan\theta + 2)$

29. $y' = \dfrac{d}{dx}\tan x = \sec^2 x$

$y'' = \dfrac{d}{dx}\sec^2 x = (2\sec x)\dfrac{d}{dx}(\sec x)$

$= (2\sec x)(\sec x \tan x)$

$= 2\sec^2 x \tan x$

30. $y' = \dfrac{d}{dx}\cot x = -\csc^2 x$

$y'' = \dfrac{d}{dx}(-\csc^2 x) = (-2\csc x)\dfrac{d}{dx}(\csc x)$

$= (-2\csc x)(-\csc x \cot x)$

$= 2\csc^2 x \cot x$

31. $y' = \dfrac{d}{dx}\cot(3x-1) = -\csc^2(3x-1)\dfrac{d}{dx}(3x-1)$

$= -3\csc^2(3x-1)$

$y'' = \dfrac{d}{dx}[-3\csc^2(3x-1)]$

$= -3[2\csc(3x-1)]\dfrac{d}{dx}\csc(3x-1)$

$= -3[2\csc(3x-1)]\cdot$

$\quad [-\csc(3x-1)\cot(3x-1)]\dfrac{d}{dx}(3x-1)$

$= -3[2\csc(3x-1)][-\csc(3x-1)\cot(3x-1)](3)$

$= 18\csc^2(3x-1)\cot(3x-1)$

32. $y' = \dfrac{d}{dx}\left[9\tan\left(\dfrac{x}{3}\right)\right] = 9\sec^2\left(\dfrac{x}{3}\right)\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$= 3\sec^2\left(\dfrac{x}{3}\right)$

$y'' = \dfrac{d}{dx}\left[3\sec^2\left(\dfrac{x}{3}\right)\right] = 3\left[2\sec\left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\sec\left(\dfrac{x}{3}\right)$

$= 6\left[\sec\left(\dfrac{x}{3}\right)\right]\left[\sec\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$= 2\sec^2\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)$

33. $f'(u) = \dfrac{d}{du}(u^5 + 1) = 5u^4$

$g'(x) = \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

$(f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = (5)\left(\dfrac{1}{2}\right) = \dfrac{5}{2}$

34. $f'(u) = \dfrac{d}{du}(1 - u^{-1}) = u^{-2} = \dfrac{1}{u^2}$

$g'(x) = \dfrac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}\dfrac{d}{dx}(1-x)$

$\quad = -(1-x)^{-2}(-1) = \dfrac{1}{(1-x)^2}$

$(f \circ g)'(-1) = f'(g(-1))g'(-1) = f'\left(\dfrac{1}{2}\right)g'(-1)$

$\quad = (4)\left(\dfrac{1}{4}\right) = 1$

35. $f'(u) = \dfrac{d}{du}\left(\cot\dfrac{\pi u}{10}\right) = -\csc^2\left(\dfrac{\pi u}{10}\right)\dfrac{d}{du}\left(\dfrac{\pi u}{10}\right)$

$\quad = -\dfrac{\pi}{10}\csc^2\left(\dfrac{\pi u}{10}\right)$

$g'(x) = \dfrac{d}{dx}(5\sqrt{x}) = \dfrac{5}{2\sqrt{x}}$

$(f \circ g)'(1) = f'(g(1))g'(1) = f'(5)g'(1)$

$\quad = -\dfrac{\pi}{10}\left[\csc^2\left(\dfrac{\pi}{2}\right)\right]\left(\dfrac{5}{2}\right)$

$\quad = -\dfrac{\pi}{10}(1)\left(\dfrac{5}{2}\right) = -\dfrac{\pi}{4}$

36. $f'(u) = \dfrac{d}{du}\left[u + (\cos u)^{-2}\right]$

$\quad = 1 - 2(\cos u)^{-3}\dfrac{d}{du}\cos u$

$\quad = 1 + \dfrac{2\sin u}{\cos^3 u}$

$g'(x) = \dfrac{d}{dx}(\pi x) = \pi$

$(f \circ g)'\left(\dfrac{1}{4}\right) = f'(g(\tfrac{1}{4}))g'\left(\dfrac{1}{4}\right)$

$\quad = f'\left(\dfrac{\pi}{4}\right)g'\left(\dfrac{1}{4}\right)$

$\quad = \left(1 + \dfrac{\dfrac{2}{\sqrt{2}}}{\left(\dfrac{1}{\sqrt{2}}\right)^3}\right)(\pi)$

$\quad = 5\pi$

37. $f'(u) = \dfrac{d}{du}\dfrac{2u}{u^2 + 1} = \dfrac{(u^2 + 1)\dfrac{d}{du}(2u) - (2u)\dfrac{d}{du}(u^2 + 1)}{(u^2 + 1)^2}$

$\quad = \dfrac{(u^2 + 1)(2) - (2u)(2u)}{(u^2 + 1)^2} = \dfrac{-2u^2 + 2}{(u^2 + 1)^2}$

$g'(x) = \dfrac{d}{dx}(10x^2 + x + 1) = 20x + 1$

$(f \circ g)'(0) = f'(g(0))g'(0) = f'(1)g'(0) = (0)(1) = 0$

38. $f'(u) = \dfrac{d}{du}\left(\dfrac{u-1}{u+1}\right)^2 = 2\left(\dfrac{u-1}{u+1}\right)\dfrac{d}{du}\left(\dfrac{u-1}{u+1}\right)$

$\quad = 2\left(\dfrac{u-1}{u+1}\right)\dfrac{(u+1)\dfrac{d}{du}(u-1) - (u-1)\dfrac{d}{du}(u+1)}{(u+1)^2}$

$\quad = 2\left(\dfrac{u-1}{u+1}\right)\dfrac{(u+1) - (u-1)}{(u+1)^2} = \dfrac{4(u-1)}{(u+1)^3}$

$g'(x) = \dfrac{d}{dx}(x^{-2} - 1) = -2x^{-3}$

$(f \circ g)'(-1) = f'(g(-1))g'(-1)$

$\quad = f'(0)g'(-1)$

$\quad = (-4)(2) = -8$

39. (a) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad = \dfrac{d}{du}(\cos u)\dfrac{d}{dx}(6x + 2)$

$\quad = (-\sin u)(6)$

$\quad = -6\sin u$

$\quad = -6\sin(6x + 2)$

(b) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad = \dfrac{d}{du}(\cos 2u)\dfrac{d}{dx}(3x + 1)$

$\quad = (-\sin 2u)(2)\cdot(3)$

$\quad = -6\sin 2u$

$\quad = -6\sin(6x + 2)$

40. (a) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad = \dfrac{d}{du}\sin(u + 1)\dfrac{d}{dx}(x^2)$

$\quad = \cos(u + 1)(1)\cdot 2x$

$\quad = 2x\cos(u + 1)$

$\quad = 2x\cos(x^2 + 1)$

(b) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$\quad = \dfrac{d}{du}(\sin u)\dfrac{d}{dx}(x^2 + 1)$

$\quad = (\cos u)(2x)$

$\quad = 2x\cos u$

$\quad = 2x\cos(x^2 + 1)$

41. $\dfrac{dx}{dt} = \dfrac{d}{dt}(2\cos t) = -2\sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(2\sin t) = 2\cos t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{2\cos t}{-2\sin t} = -\cot t$

This line passes through $\left(2\cos\dfrac{\pi}{4},\ 2\sin\dfrac{\pi}{4}\right) = (\sqrt{2},\ \sqrt{2})$

and has slope $-\cot\dfrac{\pi}{4} = -1$. Its equation is

$y = -(x - \sqrt{2}) + \sqrt{2}$, or $y = -x + 2\sqrt{2}$.

42. $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sin 2\pi t) = (\cos 2\pi t)\dfrac{d}{dt}(2\pi t) = 2\pi\cos 2\pi t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(\cos 2\pi t) = (-\sin 2\pi t)\dfrac{d}{dt}(2\pi t) = -2\pi\sin 2\pi t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t$

The line passes through $\left(\sin\dfrac{2\pi}{-6},\ \cos\dfrac{2\pi}{-6}\right) = \left(-\dfrac{\sqrt{3}}{2},\ \dfrac{1}{2}\right)$ and

has slope $-\tan\dfrac{2\pi}{-6} = \sqrt{3}$. Its equation is

$y = \sqrt{3}\left(x + \dfrac{\sqrt{3}}{2}\right) + \dfrac{1}{2}$, or $y = \sqrt{3}x + 2$.

43. $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sec^2 t - 1) = (2\sec t)\dfrac{d}{dt}(\sec t)$

$\qquad\qquad = (2\sec t)(\sec t\tan t)$

$\qquad\qquad = 2\sec^2 t\tan t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\tan t = \sec^2 t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\sec^2 t}{2\sec^2 t\tan t} = \dfrac{1}{2}\cot t.$

The line passes through

$\left(\sec^2\left(-\dfrac{\pi}{4}\right) - 1,\ \tan\left(-\dfrac{\pi}{4}\right)\right) = (1,\ -1)$ and has

slope $\dfrac{1}{2}\cot\left(-\dfrac{\pi}{4}\right) = -\dfrac{1}{2}$. Its equation

is $y = -\dfrac{1}{2}(x - 1) - 1$, or $y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

44. $\dfrac{dx}{dt} = \dfrac{d}{dt}\sec t = \sec t\tan t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\tan t = \sec^2 t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\sec^2 t}{\sec t\tan t} = \dfrac{\sec t}{\tan t} = \dfrac{1}{\sin t} = \csc t$

The line passes through $\left(\sec\dfrac{\pi}{6},\ \tan\dfrac{\pi}{6}\right) = \left(\dfrac{2}{\sqrt{3}},\ \dfrac{1}{\sqrt{3}}\right)$ and has

slope $\csc\dfrac{\pi}{6} = 2$. Its equation is $y = 2\left(x - \dfrac{2}{\sqrt{3}}\right) + \dfrac{1}{\sqrt{3}}$,

or $y = 2x - \sqrt{3}$.

45. $\dfrac{dx}{dt} = \dfrac{d}{dt}t = 1$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\sqrt{t} = \dfrac{1}{2\sqrt{t}}$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{1/(2\sqrt{t})}{1} = \dfrac{1}{2\sqrt{t}}$

The line passes through $\left(\dfrac{1}{4},\ \sqrt{\dfrac{1}{4}}\right) = \left(\dfrac{1}{4},\ \dfrac{1}{2}\right)$ and has slope

$\dfrac{1}{2\sqrt{\dfrac{1}{4}}} = 1$. Its equation is $y = 1\left(x - \dfrac{1}{4}\right) + \dfrac{1}{2}$, or $y = x + \dfrac{1}{4}$.

46. $\dfrac{dx}{dt} = \dfrac{d}{dt}(2t^2 + 3) = 4t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(t^4) = 4t^3$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{4t^3}{4t} = t^2$

The line passes through $(2(-1)^2 + 3,\ (-1)^4) = (5, 1)$ and has

slope $(-1)^2 = 1$. Its equation is $y = 1(x - 5) + 1$, or $y = x - 4$.

47. $\dfrac{dx}{dt} = \dfrac{d}{dt}(t - \sin t) = 1 - \cos t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(1 - \cos t) = \sin t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\sin t}{1 - \cos t}$

The line passes through

$\left(\dfrac{\pi}{3} - \sin\dfrac{\pi}{3},\ 1 - \cos\dfrac{\pi}{3} \right) = \left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2},\ \dfrac{1}{2} \right)$ and has slope

$\dfrac{\sin\left(\dfrac{\pi}{3}\right)}{1 - \cos\left(\dfrac{\pi}{3}\right)} = \sqrt{3}.$ Its equation is

$y = \sqrt{3}\left(x - \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2} \right) + \dfrac{1}{2},$ or

$y = \sqrt{3}x + 2 - \dfrac{\pi}{\sqrt{3}}.$

48. $\dfrac{dx}{dt} = \dfrac{d}{dt}\cos t = -\sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(1 + \sin t) = \cos t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\cos t}{-\sin t} = -\cot t$

The line passes through $\left(\cos\dfrac{\pi}{2},\ 1 + \sin\dfrac{\pi}{2} \right) = (0, 2)$ and

has slope $-\cot\left(\dfrac{\pi}{2}\right) = 0.$ Its equation is $y = 2.$

49. (a) $\dfrac{dx}{dt} = \dfrac{d}{dt}(t^2 + t) = 2t + 1$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\sin t = \cos t$

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\cos t}{2t + 1}$

(b) $\dfrac{d}{dt}\left(\dfrac{dy}{dx} \right) = \dfrac{d}{dt}\dfrac{\cos t}{2t + 1}$

$= \dfrac{(2t + 1)\dfrac{d}{dt}(\cos t) - (\cos t)\dfrac{d}{dt}(2t + 1)}{(2t + 1)^2}$

$= \dfrac{(2t + 1)(-\sin t) - (\cos t)(2)}{(2t + 1)^2}$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2}$

(c) Let $u = \dfrac{dy}{dx}.$

Then $\dfrac{du}{dt} = \dfrac{du}{dx}\dfrac{dx}{dt}$, so $\dfrac{du}{dx} = \dfrac{du}{dt} \div \dfrac{dx}{dt}.$ Therefore,

$\dfrac{d}{dx}\left(\dfrac{dy}{dx} \right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx} \right) \div \dfrac{dx}{dt}$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2} \div (2t + 1)$

$= -\dfrac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^3}$

(d) The expression in part (c).

50. Since the radius passes through $(0, 0)$ and $(2\cos t, 2\sin t)$, it has slope given by $\tan t$. But the slope of the tangent line is

$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{2\cos t}{-2\sin t} = -\cot t,$ which is the negative reciprocal

of $\tan t$. This means that the radius and the tangent line are perpendicular. (The preceding argument breaks down when

$t = \dfrac{k\pi}{2},$ where k is an integer. At these values, either the

radius is horizontal and the tangent line is vertical or the radius is vertical and the tangent line is horizontal, so the result still holds.)

51. $\dfrac{ds}{dt} = \dfrac{ds}{d\theta}\dfrac{d\theta}{dt} = \dfrac{d}{d\theta}(\cos\theta)\dfrac{d\theta}{dt}$

$= (-\sin\theta)\left(\dfrac{d\theta}{dt} \right)$

When $\theta = \dfrac{3\pi}{2}$ and $\dfrac{d\theta}{dt} = 5,$ $\dfrac{ds}{dt} = \left(-\sin\dfrac{3\pi}{2} \right)(5) = 5.$

52. $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{d}{dx}(x^2 + 7x - 5)\dfrac{dx}{dt}$

$= (2x + 7)\left(\dfrac{dx}{dt} \right)$

When $x = 1$ and $\dfrac{dx}{dt} = \dfrac{1}{3},$ $\dfrac{dy}{dt} = [2(1) + 7]\left(\dfrac{1}{3} \right) = 3.$

53. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin\dfrac{x}{2} = \left(\cos\dfrac{x}{2} \right)\dfrac{d}{dx}\left(\dfrac{x}{2} \right) = \dfrac{1}{2}\cos\dfrac{x}{2}$

Since the range of the function $f(x) = \dfrac{1}{2}\cos\dfrac{x}{2}$ is $\left[-\dfrac{1}{2}, \dfrac{1}{2} \right],$

the largest possible value of $\dfrac{dy}{dx}$ is $\dfrac{1}{2}.$

54. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin mx) = (\cos mx)\dfrac{d}{dx}(mx) = m\cos mx$

The desired line has slope $y'(0) = m\cos 0 = m$ and passes through $(0, 0)$, so its equation is $y = mx.$

55. $\dfrac{dy}{dx} = \dfrac{d}{dx} 2\tan\dfrac{\pi x}{4} = \left(2\sec^2\dfrac{\pi x}{4}\right)\dfrac{d}{dx}\left(\dfrac{\pi x}{4}\right)$

$= \dfrac{\pi}{2}\sec^2\left(\dfrac{\pi x}{4}\right)$

$y'(1) = \dfrac{\pi}{2}\sec^2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}(\sqrt{2})^2 = \pi.$

The tangent line has slope π and passes through

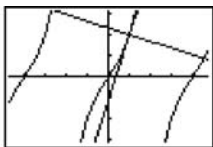$\left(1, 2\tan\dfrac{\pi}{4}\right) = (1, 2).$ Its equation is $y = \pi(x-1)+2,$ or

$y = \pi x - \pi + 2.$

The normal line has slope $-\dfrac{1}{\pi}$ and passes through $(1, 2).$

Its equation is $y = -\dfrac{1}{\pi}(x-1)+2,$ or $y = -\dfrac{1}{\pi}x + \dfrac{1}{\pi}+2.$

Graphical support:

[−4.7, 4.7] by [−3.1, 3.1]

56. (a) $\dfrac{d}{dx}[2f(x)] = 2f'(x)$

At $x = 2,$ the derivative is $2f'(2) = 2\left(\dfrac{1}{3}\right) = \dfrac{2}{3}.$

(b) $\dfrac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

At $x = 3$, the derivative is $f'(3)+g'(3) = 2\pi + 5.$

(c) $\dfrac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x)+g(x)f'(x)$

At $x = 3,$ the derivative is

$f(3)g'(3)+g(3)f'(3) = (3)(5)+(-4)(2\pi)$
$= 15 - 8\pi.$

(d) $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

At $x = 2,$ the derivative is

$\dfrac{g(2)f'(2)-f(2)g'(2)}{[g(2)]^2} = \dfrac{(2)\left(\dfrac{1}{3}\right)-(8)(-3)}{(2)^2}$

$= \dfrac{\dfrac{74}{3}}{4} = \dfrac{37}{6}.$

(e) $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 2,$ the derivative is

$f'(g(2))g'(2) = f'(2)g'(2) = \left(\dfrac{1}{3}\right)(-3) = -1.$

(f) $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2\sqrt{f(x)}}\dfrac{d}{dx}f(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

At $x = 2,$ the derivative is

$\dfrac{f'(2)}{2\sqrt{f(2)}} = \dfrac{\dfrac{1}{3}}{2\sqrt{8}} = \dfrac{1}{6(2\sqrt{2})} = \dfrac{1}{12\sqrt{2}}.$

(g) $\dfrac{d}{dx}\dfrac{1}{g^2(x)} = \dfrac{d}{dx}[g(x)]^{-2} = -2[g(x)]^{-3}\dfrac{d}{dx}g(x) = -\dfrac{2g'(x)}{[g(x)]^3}$

At $x = 3,$ the derivative is

$-\dfrac{2g'(3)}{[g(3)]^3} = -\dfrac{2(5)}{(-4)^3} = -\dfrac{10}{-64} = \dfrac{5}{32}.$

(h) $\dfrac{d}{dx}\sqrt{f^2(x)+g^2(x)} = \dfrac{1}{2\sqrt{f^2(x)+g^2(x)}}\dfrac{d}{dx}[f^2(x)+g^2(x)]$

$= \dfrac{1}{2\sqrt{f^2(x)+g^2(x)}}[2f(x)\dfrac{d}{dx}f(x)+2g(x)\dfrac{d}{dx}g(x)]$

$= \dfrac{f(x)f'(x)+g(x)g'(x)}{\sqrt{f^2(x)+g^2(x)}}$

At $x = 2,$ the derivative is

$\dfrac{f(2)f'(2)+g(2)g'(2)}{\sqrt{f^2(2)+g^2(2)}} = \dfrac{(8)\left(\dfrac{1}{3}\right)+(2)(-3)}{\sqrt{8^2+2^2}}$

$= \dfrac{-\dfrac{10}{3}}{\sqrt{68}} = -\dfrac{\dfrac{10}{3}}{2\sqrt{17}} = -\dfrac{5}{3\sqrt{17}}$

57. $\dfrac{d}{dx}\cos(x°) = \dfrac{d}{dx}\cos\left(\dfrac{\pi x}{180}\right) = -\dfrac{\pi}{180}\sin\left(\dfrac{\pi x}{180}\right) = -\dfrac{\pi}{180}\sin(x°)$

58. (a) $\dfrac{d}{dx}[5f(x)-g(x)] = 5f'(x)-g'(x)$

At $x = 1,$ the derivative is

$5f'(1)-g'(1) = 5\left(-\dfrac{1}{3}\right)-\left(-\dfrac{8}{3}\right) = 1.$

(b) $\dfrac{d}{dx}f(x)g^3(x) = f(x)\dfrac{d}{dx}g^3(x)+g^3(x)\dfrac{d}{dx}f(x)$

$= f(x)[3g^2(x)]\dfrac{d}{dx}g(x)+g^3(x)f'(x)$

$= 3f(x)g^2(x)g'(x)+g^3(x)f'(x)$

At $x = 0,$ the derivative is $3f(0)g^2(0)g'(0)+g^3(0)f'(0)$

$= 3(1)(1)^2\left(\dfrac{1}{3}\right)+(1)^3(5) = 6.$

58. Continued

(c) $\dfrac{d}{dx}\dfrac{f(x)}{g(x)+1} = \dfrac{[g(x)+1]\dfrac{d}{dx}f(x)-f(x)\dfrac{d}{dx}[g(x)+1]}{[g(x)+1]^2}$

$\qquad = \dfrac{[g(x)+1]f'(x)-f(x)g'(x)}{[g(x)+1]^2}$

At $x = 1$, the derivative is

$\dfrac{[g(1)+1]f'(1)-f(1)g'(1)}{[g(1)+1]^2} = \dfrac{(-4+1)\left(-\dfrac{1}{3}\right)-(3)\left(-\dfrac{8}{3}\right)}{(-4+1)^2}$

$\qquad = \dfrac{9}{9} = 1.$

(d) $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 0$, the derivative is

$f'(g(0))g'(0) = f'(1)g'(0) = \left(-\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right) = -\dfrac{1}{9}.$

(e) $\dfrac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = 0$, the derivative is

$g'(f(0))f'(0) = g'(1)f'(0) = \left(-\dfrac{8}{3}\right)(5) = -\dfrac{40}{3}$

(f) $\dfrac{d}{dx}[g(x)+f(x)]^{-2} = -2[g(x)+f(x)]^{-3}\dfrac{d}{dx}[g(x)+f(x)]$

$\qquad = -\dfrac{2[g'(x)+f'(x)]}{[g(x)+f(x)]^3}$

At $x = 1$, the derivative is

$-\dfrac{2[g'(1)+f'(1)]}{[g(1)+f(1)]^3} = -\dfrac{2\left(-\dfrac{8}{3}-\dfrac{1}{3}\right)}{(-4+3)^3} = -\dfrac{-6}{-1} = -6.$

(g) $\dfrac{d}{dx}[f(x+g(x))] = f'(x+g(x))\dfrac{d}{dx}[x+g(x)]$

$\qquad = f'(x+g(x))(1+g'(x))$

At $x = 0$, the derivative is

$f'(0+g(0))(1+g'(0)) = f'(0+1)\left(1+\dfrac{1}{3}\right)$

$\qquad = f'(1)\left(\dfrac{4}{3}\right)$

$\qquad = \left(-\dfrac{1}{3}\right)\left(\dfrac{4}{3}\right) = -\dfrac{4}{9}.$

59. For $y = \sin 2x$, $y' = (\cos 2x)\dfrac{d}{dx}(2x) = 2\cos 2x$ and the slope

at the origin is 2.

For $y = -\sin\dfrac{x}{2}$, $y' = \left(-\cos\dfrac{x}{2}\right)\dfrac{d}{dx}\left(\dfrac{x}{2}\right) = -\dfrac{1}{2}\cos\dfrac{x}{2}$ and the

slope at the origin is $-\dfrac{1}{2}$. Since the slopes of the two

tangent lines are 2 and $-\dfrac{1}{2}$, the lines are perpendicular and

the curves are orthogonal.

A graph of the two curves along with the tangents $y = 2x$

and $y = -\dfrac{1}{2}x$ is shown.
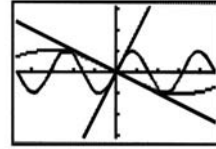
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

60. Because the symbols $\dfrac{dy}{dx}, \dfrac{dy}{du}$, and $\dfrac{du}{dx}$ are not fractions.

The individual symbols dy, dx, and du do not have
numerical values.

61. Velocity: $s'(t) = -2\pi bA \sin(2\pi bt)$

acceleration: $s''(t) = -4\pi^2 b^2A \cos(2\pi bt)$

jerk: $s'''(t) = 8\pi^3 b^3A \sin(2\pi bt)$

The velocity, amplitude, and jerk and proportional to b,
b^2, and b^3 respectively. If the frequency b is doubled, then
the amplitude of the velocity is doubled, the amplitude of
the acceleration is quadrupled, and the amplitude of the jerk
is multiplied by 8.

62. (a) $y'(t) = \dfrac{d}{dt}37\sin\left[\dfrac{2\pi}{365}(x-101)\right]+\dfrac{d}{dt}(25)$

$\qquad = 37\cos\left[\dfrac{2\pi}{365}(x-101)\right]\cdot\dfrac{d}{dx}\left[\dfrac{2\pi}{365}(x-101)\right]+0$

$\qquad = 37\cos\left[\dfrac{2\pi}{365}(x-101)\right]\cdot\dfrac{2\pi}{365}$

$\qquad = \dfrac{74\pi}{365}\cos\left[\dfrac{2\pi}{365}(x-101)\right]$

Since $\cos u$ is greatest when $u = 0$, $\pm 2\pi$, and so on,

$y'(t)$ is greatest when $\dfrac{2\pi}{365}(x-101) = 0$, or

$x = 101$. The temperature is increasing the fastest on
day 101 (April 11).

(b) The rate of increase is

$y'(101) = \dfrac{74\pi}{365} \approx 0.637$ degrees per day.

63. Velocity: $s'(t) = \dfrac{d}{dt}\sqrt{1+4t} = \dfrac{1}{2\sqrt{1+4t}}\dfrac{d}{dt}(1+4t)$

$\qquad = \dfrac{4}{2\sqrt{1+4t}} = \dfrac{2}{\sqrt{1+4t}}$

At $t = 6$, the velocity is $\dfrac{2}{\sqrt{1+4(6)}} = \dfrac{2}{5}$ m/sec

63. Continued

Acceleration: $s''(t) = \dfrac{d}{dt} \dfrac{2}{\sqrt{1+4t}}$

$= \dfrac{(\sqrt{1+4t})\dfrac{d}{dt}(2) - 2\dfrac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$

$= \dfrac{-2\left(\dfrac{1}{2\sqrt{1+4t}}\right)\dfrac{d}{dt}(1+4t)}{1+4t}$

$= \dfrac{\dfrac{-4}{\sqrt{1+4t}}}{1+4t} = -\dfrac{4}{(1+4t)^{3/2}}$

At $t = 6$, the acceleration is $-\dfrac{4}{[1+4(6)]^{3/2}} = -\dfrac{4}{125}\,\text{m/sec}^2$

64. Acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \left(\dfrac{dv}{ds}\right)(v) = \left[\dfrac{d}{ds}(k\sqrt{s})\right](k\sqrt{s})$

$= \left(\dfrac{k}{2\sqrt{s}}\right)(k\sqrt{s}) = \dfrac{k^2}{2}$, a constant.

65. Note that this Exercise concerns itself with the slowing down caused by the earth's atmosphere, *not* the acceleration caused by gravity.

Given: $v = \dfrac{k}{\sqrt{s}}$

Acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \left(\dfrac{dv}{ds}\right)(v) = (v)\left(\dfrac{dv}{ds}\right)$

$= \left(\dfrac{k}{\sqrt{s}}\right)\dfrac{d}{ds}\dfrac{k}{\sqrt{s}}$

$= \left(\dfrac{k}{\sqrt{s}}\right)\left(\dfrac{\sqrt{s}\dfrac{d}{ds}(k) - k\dfrac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right)$

$= \left(\dfrac{k}{\sqrt{s}}\right)\left(\dfrac{\dfrac{-k}{(2\sqrt{s})}}{s}\right)$

$= -\dfrac{k^2}{2s^2}, s \geq 0$

Thus, the acceleration is inversely proportional to s^2.

66. Acceleration $= \dfrac{dv}{dt} = \dfrac{df(x)}{dt} = \dfrac{df(x)}{dx}\dfrac{dx}{dt} = f'(x)f(x)$

67. $\dfrac{dT}{du} = \dfrac{dT}{dL}\dfrac{dL}{du} = \left(\dfrac{d}{dL}2\pi\sqrt{\dfrac{L}{g}}\right)(kL)$

$= \left(2\pi\dfrac{1}{2\sqrt{\dfrac{L}{g}}}\right)\left(\dfrac{d}{dL}\dfrac{L}{g}\right)(kL)$

$= \left(\dfrac{\pi}{\sqrt{\dfrac{L}{g}}}\right)\left(\dfrac{1}{g}\right)(kL) = k\pi\sqrt{\dfrac{L}{g}} = \dfrac{kT}{2}$

68. No, this does not contradict the Chain Rule. The Chain Rule states that if two functions are differentiable at the appropriate points, then their composite must also be differentiable. It does not say: If a composite is differentiable, then the functions which make up the composite must all be differentiable.

69. Yes. Note that $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

70. False. See example 8.

71. False. It is $+1$.

72. E. $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan(4x)$

$y = \tan u \qquad u = 4x$

$\dfrac{dy}{du} = \sec^2 u \qquad \dfrac{du}{dx} = 4$

$\dfrac{dy}{dx} = 4\sec^2(4x)$

73. C. $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos^2(x^3 + x^2)$

$y = \cos^2 u \qquad\qquad u = x^3 + x^2$

$\dfrac{dy}{du} = -2\sin u \cos u \qquad \dfrac{du}{dx} = 3x^2 + 2x$

$\dfrac{dy}{dx} = -2(3x^2 + 2x)\cos(3x^2 + 2x)\sin(3x^2 + 2x)$

74. A. $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dy}{dt}}$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(-1 + \sin t)$

$= \cos t$

$\dfrac{dx}{dt} = \dfrac{d}{dt}(t - \cos t)$

$= 1 + \sin t$

$\dfrac{\dfrac{dy}{dt}}{\dfrac{dy}{dt}} = \dfrac{\cos t}{1 + \sin t}$

$\dfrac{dy}{dx}\bigg|_{t=0} = \dfrac{\cos 0}{1 + \sin 0} = 1$

$x(0) = 0 - \cos 0 = -1$

$y(0) = -1 + \sin 0 = -1$

$y = 1(x - (-1)) + (-1)) = x$

75. B. See problem 74.

$$\frac{dy}{dx} = \frac{\cos t}{1 + \sin t} = 0$$

$$\cos t = 0$$

$$t = \frac{\pi}{2}$$

76. For $h = 1$:
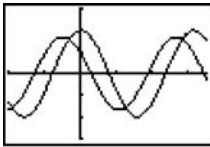
[−2, 3.5] by [−3, 3]

For $h = 0.5$:
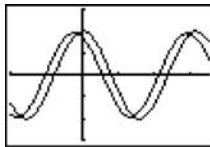
[−2, 3.5] by [−3, 3]

For $h = 0.2$:
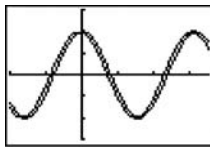
[−2, 3.5] by [−3, 3]

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = 2 \cos 2x$). This is because $2 \cos 2x$ is the derivative of $\sin 2x$, and the second curve is the difference quotient used to define the derivative of $\sin 2x$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

77. For $h = 1$:
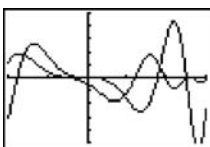
[−2, 3] by [−5, 5]

For $h = 0.3$:
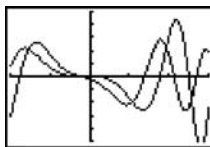
[−2, 3] by [−5, 5]

For $h = 0.3$:
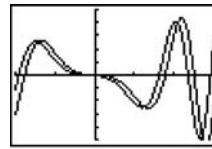
[−2, 3] by [−5, 5]

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x \sin(x^2)$). This is because $-2x \sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

78. (a) Let $f(x) = |x|$,

Then

$$\frac{d}{dx}|u| = \frac{d}{dx}f(u) = f'(u)\frac{du}{dx} = \left(\frac{d}{dx}|u|\right)\left(\frac{du}{dx}\right) = \frac{u}{|u|}u'.$$

The derivative of the absolute value function is $+1$ for positive values, -1 for negative values, and undefined at 0. So $f'(u) = \begin{cases} -1, & u < 0 \\ 1, & u > 0. \end{cases}$

But this is exactly how the expression $\dfrac{u}{|u|}$ evaluates.

(b) $f'(x) = \left[\dfrac{d}{dx}(x^2 - 9)\right] \cdot \dfrac{x^2 - 9}{|x^2 - 9|} = \dfrac{(2x)(x^2 - 9)}{|x^2 - 9|}$

$$g'(x) = \frac{d}{dx}(|x|\sin x)$$

$$= |x|\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}|x|$$

$$= |x|\cos x + \frac{x \sin x}{|x|}$$

Note: The expression for $g'(x)$ above is undefined at $x = 0$, but actually

$$g'(0) = \lim_{h \to 0}\frac{g(0+h) - g(0)}{h} = \lim_{h \to 0}\frac{|h|\sin h}{h} = 0.$$

Therefore, we may express the derivative as

$$g'(x) = \begin{cases} |x|\cos x + \dfrac{x \sin x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

79. $\dfrac{dG}{dx} = \dfrac{d}{dx}\sqrt{uv} = \dfrac{d}{dx}\sqrt{x(x+c)} = \dfrac{d}{dx}\sqrt{x^2 + cx}$

$$= \frac{1}{2\sqrt{x^2 + cx}}\frac{d}{dx}(x^2 + cx) = \frac{2x + c}{2\sqrt{x^2 + cx}} = \frac{x + (x + c)}{2\sqrt{x(x+c)}}$$

$$= \frac{u + v}{2\sqrt{uv}} = \frac{A}{G}$$

Quick Quiz Sections 3.4-3.6

1. B. $y = \sin^4 u$ $\qquad u = 3x$

$$\frac{dy}{du} = 4\sin^3 u \cos u \qquad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = 12\sin^3(3x)\cos(3x)$$

2. A. $\quad y = \cos x + \tan x$

$\qquad y' = -\sin x + \sec^2 x$

$\qquad y'' = -\cos x + 2\sec^2 x \tan x$

3. C. $x = 3\sin t \qquad y = 2\cos t$

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$$

$$\frac{dy}{dt} = \frac{d}{dt}(2\cos t) = -2\sin t$$

$$\frac{dx}{dt} = \frac{d}{dt}(3\sin t) = 3\cos t$$

$$\frac{dy}{dx} = \frac{-2\sin t}{3\cos t} = -\frac{2}{3}\tan t.$$

4. (a) $s(0) = -(0)^2 + (0) + 2$

$\qquad s(0) = 2$

$\qquad (0, 2)$

(b) $-t^2 + t + 2 > 0$

$\qquad (t+1)(-t+2) > 0$

$\qquad -1 < t < \dfrac{1}{2}$ but $t < 0$ not real

$\qquad 0 \le t < \dfrac{1}{2}$

(c) $t > \dfrac{1}{2}$

(d) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(-t^2 + t + 2)$

$\qquad\qquad = -2t + 1$

(e) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(-2t + 1)$

$\qquad\qquad = -2$

(f) $t = \dfrac{1}{2}$

Section 3.7 Implicit Differentiation
(pp. 157–164)

Exploration 1   An Unexpected Derivative

1. $2x - 2y - 2xy' + 2yy' = 0$. Solving for y', we find that

$$\frac{dy}{dx} = 1 \text{ (provided } y \ne x\text{)}.$$

2. With a constant derivative of 1, the graph would seem to be a line with slope 1.

3. Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.

4. Factoring the original equation, we have

$$[(x - y) - 2][(x - y) + 2] = 0$$
$$\therefore x - y - 2 = 0 \text{ or } x - y + 2 = 0$$
$$\therefore y = x - 2 \text{ or } y = x + 2.$$

The graph is shown below.

[−4.7, 4.7] by [−3.1, 3.1]

5. At each point (x, y) on either line, $\dfrac{dy}{dx} = 1$. The condition $y \ne x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on x or y, but there are no inconsistencies.

Quick Review 3.7

1. $x - y^2 = 0$

$$x = y^2$$
$$\pm\sqrt{x} = y$$
$$y_1 = \sqrt{x},\ y_2 = -\sqrt{x}$$
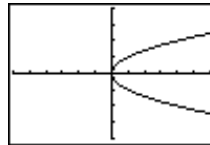
[-6, 6] by [-4, 4]

2. $4x^2 + 9y^2 = 36$

$$9y^2 = 36 - 4x^2$$
$$y^2 = \frac{36 - 4x^2}{9} = \frac{4}{9}(9 - x^2)$$
$$y = \pm\frac{2}{3}\sqrt{9 - x^2}$$
$$y_1 = \frac{2}{3}\sqrt{9 - x^2}, \ y_2 = -\frac{2}{3}\sqrt{9 - x^2}$$
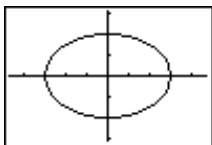
[-4.7, 4.7] by [-3.1, 3.1]

3. $\qquad x^2 - 4y^2 = 0$

$$(x + 2y)(x - 2y) = 0$$
$$y = \pm\frac{x}{2}$$
$$y_1 = \frac{x}{2}, \ y_2 = -\frac{x}{2}$$
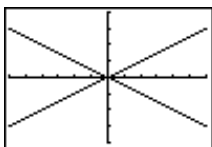
[-6, 6] by [-4, 4]

4. $x^2 + y^2 = 9$

$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$
$$y_1 = \sqrt{9 - x^2}, \ y_2 = -\sqrt{9 - y^2}$$
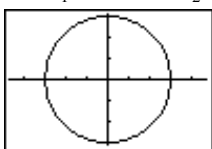
[-4.7, 4.7] by [-3.1, 3.1]

5. $x^2 + y^2 = 2x + 3$

$$y^2 = 2x + 3 - x^2$$
$$y = \pm\sqrt{2x + 3 - x^2}$$
$$y_1 = \sqrt{2x + 3 - x^2}, \ y^2 = -\sqrt{2x + 3 - x^2}$$
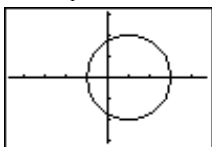
[-4.7, 4.7] by [-3.1, 3.1]

6. $x^2 y' - 2xy = 4x - y$

$$x^2 y' = 4x - y + 2xy$$
$$y' = \frac{4x - y + 2xy}{x^2}$$

7. $y' \sin x - x \cos x = xy' + y$

$$y' \sin x - xy' = y + x\cos x$$
$$(\sin x - x)y' = y + x\cos x$$
$$y' = \frac{y + x\cos x}{\sin x - x}$$

8. $x(y^2 - y') = y'(x^2 - y)$

$$xy^2 = y'(x^2 - y + x)$$
$$y' = \frac{xy^2}{x^2 - y + x}$$

9. $\sqrt{x}(x - \sqrt[3]{x} = x^{1/2}(x - x^{1/3})$

$$= x^{1/2}x - x^{1/2}x^{1/3}$$
$$= x^{3/2} - x^{5/6}$$

10. $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} = \dfrac{x + x^{2/3}}{x^{3/2}}$

$$= \frac{x}{x^{3/2}} + \frac{x^{2/3}}{x^{3/2}}$$
$$= x^{-1/2} + x^{-5/6}$$

Section 3.7 Exercises

1. $\qquad\qquad x^2 y + xy^2 = 6$

$$\frac{d}{dx}(x^2 y) + \frac{d}{dx}(xy^2) = \frac{d}{dx}(6)$$
$$x^2\frac{dy}{dx} + y(2x) + x(2y)\frac{dy}{dx} + y^2(1) = 0$$
$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -(2xy + y^2)$$
$$(2xy + x^2)\frac{dy}{dx} = -(2xy + y^2)$$
$$\frac{dy}{dx} = -\frac{2xy + y^2}{2xy + x^2}$$

2. $\qquad\qquad x^3 + y^3 = 18xy$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(18xy)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 18x\frac{dy}{dx} + 18y(1)$$
$$3y^2\frac{dy}{dx} - 18x\frac{dy}{dx} = 18y - 3x^2$$
$$(3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2$$
$$\frac{dy}{dx} = \frac{18y - 3x^2}{3y^2 - 18x}$$
$$\frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$$

3. $y^2 = \dfrac{x-1}{x+1}$

$\dfrac{d}{dx}y^2 = \dfrac{d}{dx}\dfrac{x-1}{x+1}$

$2y\dfrac{dy}{dx} = \dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$2y\dfrac{dy}{dx} = \dfrac{2}{(x+1)^2}$

$\dfrac{dy}{dx} = \dfrac{1}{y(x+1)^2}$

4. $x^2 = \dfrac{x-y}{x+y}$

$\dfrac{d}{dx}(x^2) = \dfrac{d}{dx}\dfrac{x-y}{x+y}$

$2x = \dfrac{(x+y)\left(1-\dfrac{dy}{dx}\right)-(x-y)\left(1+\dfrac{dy}{dx}\right)}{(x+y)^2}$

$2x = \dfrac{\left[x-x\dfrac{dy}{dx}+y-y\dfrac{dy}{dx}\right]-\left[x+x\dfrac{dy}{dx}-y-y\dfrac{dy}{dx}\right]}{(x+y)^2}$

$2x = \dfrac{2y-2x\dfrac{dy}{dx}}{(x+y)^2}$

$x(x+y)^2 = y - x\dfrac{dy}{dx}$

$x\dfrac{dy}{dx} = y - x(x+y)^2$

$\dfrac{dy}{dx} = \dfrac{y-x(x+y)^2}{x} = \dfrac{y}{x}-(x+y)^2$

Alternate solution:

$x^2 = \dfrac{x-y}{x+y}$

$x^2(x+y) = x-y$

$x^3 + x^2 y = x - y$

$\dfrac{d}{dx}(x^3)+\dfrac{d}{dx}(x^2 y) = \dfrac{d}{dx}(x)-\dfrac{d}{dx}(y)$

$3x^2 + x^2\dfrac{dy}{dx}+y(2x) = 1-\dfrac{dy}{dx}$

$(x^2+1)\dfrac{dy}{dx} = 1-3x^2-2xy$

$\dfrac{dy}{dx} = \dfrac{1-3x^2-2xy}{x^2+1}$

5. $x = \tan y$

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\tan y)$

$1 = \sec^2 y\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \cos^2 y$

6. $x = \sin y$

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\sin y)$

$1 = \cos y\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \sec y$

7. $x + \tan xy = 0$

$\dfrac{d}{dx}(x)+\dfrac{d}{dx}(\tan xy) = \dfrac{d}{dx}(0)$

$1+\sec^2(xy)\dfrac{d}{dx}(xy) = 0$

$1+(\sec^2 xy)[x\dfrac{dy}{dx}+(y)\,(1)] = 0$

$(\sec^2 xy)\,(x)\dfrac{dy}{dx} = -1-(\sec^2 xy)(y)$

$\dfrac{dy}{dx} = \dfrac{-1-y\sec^2 xy}{x\sec^2 xy}$

$\dfrac{dy}{dx} = -\dfrac{1}{x}\cos^2 xy - \dfrac{y}{x}$

8. $x + \sin y = xy$

$\dfrac{d}{dx}(x)+\dfrac{d}{dx}(\sin y) = \dfrac{d}{dx}(xy)$

$1+(\cos y)\dfrac{dy}{dx} = x\dfrac{dy}{dx}+(y)(1)$

$(\cos y - x)\dfrac{dy}{dx} = -1+y$

$\dfrac{dy}{dx} = \dfrac{-1+y}{\cos y - x} = \dfrac{1-y}{x-\cos y}$

9. $\dfrac{d}{dx}(x^2+y^2) = \dfrac{d}{dx}(13)$

$2x+2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{y}, \ -\dfrac{-2}{3} = \dfrac{2}{3}$

10. $\dfrac{d}{dx}(x^2 + y^2) = \dfrac{d}{dx}(9)$

$2x + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad -\dfrac{0}{3} = 0$

11. $\dfrac{d}{dx}((x-1)^2 + (y-1)^2) = \dfrac{d}{dx}(13)$

$2(x-1)1 + (2(y-1)1)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x-1}{y-1}, \quad -\dfrac{3-1}{4-1} = -\dfrac{2}{3}$

12. $\dfrac{d}{dx}((x+2)^2 + (y+3)^2) = \dfrac{d}{dx}25$

$2(x+2)1 + (2(y+3)1)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x+2}{y+3}, \quad -\dfrac{1+2}{-7+3} = \dfrac{3}{4}$

13. $\dfrac{d}{dx}(x^2 y - xy^2) = \dfrac{d}{dx}(4)$

$2xy - y^2 + (2xy - x^2)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2xy - y^2}{2xy - x^2},$

defined at every point except where $x = 0$ or $y = \dfrac{x}{2}$.

14. $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(\cos y)$

$1 = -\sin y\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y},$

defined everywhere.

15. $\dfrac{d}{dx}(x^3 + y^3) = \dfrac{d}{dx}(xy)$

$3x^2 + 3y^2\dfrac{dy}{dx} = y + x\dfrac{dx}{dy}$

$3x^2 - y = (x - 3y^2)\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{3x^2 - y}{x - 3y^2},$

defined everywhere except where $y^2 = \dfrac{x}{3}$

16. $\dfrac{d}{dx}(x^2 + 4xy + 4y^2 - 3x) = \dfrac{d}{dx}(6)$

$2x + 4y - 3 + (4x + 8y)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{3 - 2x - 4y}{4x + 8y},$

defined everywhere except where $y = -\dfrac{1}{2}x$

17. $x^2 + xy - y^2 = 1$

$\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(xy) - \dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(1)$

$2x + x\dfrac{dy}{dx} + (y)(1) - 2y\dfrac{dy}{dx} = 0$

$(x - 2y)\dfrac{dy}{dx} = -2x - y$

$\dfrac{dy}{dx} = \dfrac{-2x - y}{x - 2y} = \dfrac{2x + y}{2y - x}$

Slope at (2, 3): $\dfrac{2(2) + 3}{2(1) - 2} = \dfrac{7}{4}$

(a) Tangent: $y = \dfrac{7}{4}(x - 2) + 3$ or $y = \dfrac{7}{4}x - \dfrac{1}{2}$

(b) Normal: $y = -\dfrac{4}{7}(x - 2) + 3$ or $y = -\dfrac{4}{7}x + \dfrac{29}{7}$

18. $x^2 + y^2 = 25$

$\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(25)$

$2x + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{y}$

Slope at $(3, -4): -\dfrac{3}{-4} = \dfrac{3}{4}$

(a) Tangent: $y = \dfrac{3}{4}(x - 3) + (-4)$ or $y = \dfrac{3}{4}x - \dfrac{25}{4}$

(b) Normal: $y = -\dfrac{4}{3}(x - 3) + (-4)$ or $y = -\dfrac{4}{3}x$

19. $x^2 y^2 = 9$

$\dfrac{d}{dx}(x^2 y^2) = \dfrac{d}{dx}(9)$

$(x^2)(2y)\dfrac{dy}{dx} + (y^2)(2x) = 0$

$2x^2 y\dfrac{dy}{dx} = -2xy^2$

$\dfrac{dy}{dx} = -\dfrac{2xy^2}{2x^2 y} = -\dfrac{y}{x}$

Slope at $(-1, 3): -\dfrac{3}{-1} = 3$

(a) Tangent: $y = 3(x + 1) + 3$ or $y = 3x + 6$

(b) Normal: $y = -\dfrac{1}{3}(x + 1) + 3$ or $y = -\dfrac{1}{3}x + \dfrac{8}{3}$

20.
$$y^2 - 2x - 4y - 1 = 0$$
$$\frac{d}{dx}(y^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(4y) - \frac{d}{dx}(1) = \frac{d}{dx}(0)$$
$$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} - 0 = 0$$
$$(2y - 4)\frac{dy}{dx} = 2$$
$$\frac{dy}{dx} = \frac{1}{y-2}$$

Slope at $(-2, 1)$: $\dfrac{1}{1-2} = -1$

(a) Tangent: $y = -(x+2) + 1$ or $y = -x - 1$

(b) Normal: $y = 1(x+2) + 1$ or $y = x + 3$

21.
$$6x^2 + 3xy + 2y^2 + 17y - 6 = 0$$
$$\frac{d}{dx}(6x^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(17y) - \frac{d}{dx}(6) = \frac{d}{dx}(0)$$
$$12x + 3x\frac{dy}{dx} + (3y)(1) + 4y\frac{dy}{dx} + 17\frac{dy}{dx} - 0 = 0$$
$$3x\frac{dy}{dx} + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = -12x - 3y$$
$$(3x + 4y + 17)\frac{dy}{dx} = -12x - 3y$$
$$\frac{dy}{dx} = \frac{-12x - 3y}{3x + 4y + 17}$$

Slope at $(-1, 0)$: $\dfrac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17} = \dfrac{12}{14} = \dfrac{6}{7}$

(a) Tangent: $y = \dfrac{6}{7}(x+1) + 0$ or $y = \dfrac{6}{7}x + \dfrac{6}{7}$

(b) Normal: $y = -\dfrac{7}{6}(x+1) + 0$ or $y = -\dfrac{7}{6}x - \dfrac{7}{6}$

22.
$$x^2 - \sqrt{3}xy + 2y^2 = 5$$
$$\frac{d}{dx}(x^2) - \sqrt{3}\frac{d}{dx}(xy) + 2\frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$
$$2x - \sqrt{3}(x)\frac{dy}{dx} - \sqrt{3}(y)(1) + 4y\frac{dy}{dx} = 0$$
$$(-x\sqrt{3} + 4y)\frac{dy}{dx} = y\sqrt{3} - 2x$$
$$\frac{dy}{dx} = \frac{y\sqrt{3} - 2x}{-x\sqrt{3} + 4y}$$

Slope at $(\sqrt{3}, 2)$: $\dfrac{2\sqrt{3} - 2\sqrt{3}}{-\sqrt{3}\sqrt{3} + 4(2)} = 0$

(a) Tangent: $y = 2$

(b) Normal: $x = \sqrt{3}$

23.
$$2xy + \pi \sin y = 2\pi$$
$$2\frac{d}{dx}(xy) + \pi\frac{d}{dx}(\sin y) = \frac{d}{dx}(2\pi)$$
$$2x\frac{dy}{dx} + 2y(1) + \pi\cos y\frac{dy}{dx} = 0$$
$$(2x + \pi\cos y)\frac{dy}{dx} = -2y$$
$$\frac{dy}{dx} = -\frac{2y}{2x + \pi\cos y}$$

Slope at $\left(1, \dfrac{\pi}{2}\right)$: $-\dfrac{2(\pi/2)}{2(1) + \pi\cos(\pi/2)} = -\dfrac{\pi}{2}$

(a) Tangent: $y = -\dfrac{\pi}{2}(x-1) + \dfrac{\pi}{2}$ or $y = -\dfrac{\pi}{2}x + \pi$

(b) Normal: $y = \dfrac{2}{\pi}(x-1) + \dfrac{\pi}{2}$ or $y = \dfrac{2}{\pi}x - \dfrac{2}{\pi} + \dfrac{\pi}{2}$

24. $x\sin 2y = y\cos 2x$
$$\frac{d}{dx}(x\sin 2y) = \frac{d}{dy}(y\cos 2x)$$
$$(x)(\cos 2y)(2)\frac{dy}{dx} + (\sin 2y)(1) =$$
$$(y)(-\sin 2x)(2) + (\cos 2x)\left(\frac{dy}{dx}\right)$$
$$(2x\cos 2y)\frac{dy}{dx} - (\cos 2x)\frac{dy}{dx} = -2y\sin 2x - \sin 2y$$
$$\frac{dy}{dx} = -\frac{2y\sin 2x + \sin 2y}{2x\cos 2y - \cos 2x}$$

Slope at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$: $-\dfrac{2\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi}{2}\right) + \sin(\pi)}{2\left(\dfrac{\pi}{4}\right)\cos(\pi) - \cos\left(\dfrac{\pi}{2}\right)}$

$$= -\frac{(\pi)(1) + 0}{\left(\dfrac{\pi}{2}\right)(-1) - 0} = 2$$

(a) Tangent: $y = 2\left(x - \dfrac{\pi}{4}\right) + \dfrac{\pi}{2}$ or $y = 2x$

(b) Normal: $y = -\dfrac{1}{2}\left(x - \dfrac{\pi}{4}\right) + \dfrac{\pi}{2}$ or $y = -\dfrac{1}{2}x + \dfrac{5\pi}{8}$

25.
$$y = 2\sin(\pi x - y)$$
$$\frac{dy}{dx} = \frac{d}{dx}2\sin(\pi x - y)$$
$$\frac{dy}{dx} = 2\cos(\pi x - y)\left(\pi - \frac{dy}{dx}\right)$$
$$[1 + 2\cos(\pi x - y)]\frac{dy}{dx} = 2\pi\cos(\pi x - y)$$
$$\frac{dy}{dx} = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}$$

Slope at $(1, 0)$: $\dfrac{2\pi\cos\pi}{1 + 2\cos\pi} = \dfrac{2\pi(-1)}{1 + 2(-1)} = 2\pi$

(a) Tangent: $y = 2\pi(x-1) + 0$ or $y = 2\pi x - 2\pi$

(b) Normal: $y = -\dfrac{1}{2\pi}(x-1) + 0$ or $y = -\dfrac{x}{2\pi} + \dfrac{1}{2\pi}$

26. $x^2 \cos^2 y - \sin y = 0$

$$\frac{d}{dx}(x^2 \cos^2 y) - \frac{d}{dx}(\sin y) = \frac{d}{dx}(0)$$

$$(x^2)(2\cos y)(-\sin y)\left(\frac{dy}{dx}\right) + (\cos^2 y)(2x) - (\cos y)\frac{dy}{dx} = 0$$

$$-(2x^2 \cos y \sin y + \cos y)\frac{dy}{dx} = -2x\cos^2 y$$

$$\frac{dy}{dx} = \frac{2x\cos^2 y}{\cos y + 2x^2 \cos y \sin y} = \frac{2x\cos y}{1 + 2x^2 \sin y}$$

Slope at $(0, \pi)$: $\dfrac{2(0)\cos\pi}{1 + 2(0)^2 \sin\pi} = 0$

(a) Tangent: $y = \pi$

(b) Normal: $x = 0$

27. $x^2 + y^2 = 1$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$
$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = -\frac{x}{y}$$
$$y'' = \frac{d}{dx}\left(-\frac{x}{y}\right)$$
$$= -\frac{(y)(1) - (x)(y')}{y^2}$$
$$= -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2}$$
$$= -\frac{x^2 + y^2}{y^3}$$

Since our original equation was $x^2 + y^2 = 1$, we may

substitute 1 for $x^2 + y^2$, giving $y'' = -\dfrac{1}{y^3}$.

28. $x^{2/3} + y^{2/3} = 1$

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(1)$$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$
$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{1/3}\right]$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{xy' - (y)(1)}{x^2}$$

$$= -\frac{1}{3}\frac{-(x)\left(\dfrac{y}{x}\right)^{1/3} - y}{x^{4/3}y^{2/3}}$$

$$= \frac{1}{3}\frac{x^{2/3}y^{1/3} + y}{x^{4/3}y^{2/3}}$$

$$= \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}}$$

Since our original equation was $x^{2/3} + y^{2/3} = 1$, we may

substitute 1 for $x^{2/3} + y^{2/3}$, giving $y'' = \dfrac{1}{3x^{4/3}y^{1/3}}$.

29. $y^2 = x^2 + 2x$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x)$$
$$2yy' = 2x + 2$$
$$y' = \frac{2x + 2}{2y} = \frac{x + 1}{y}$$
$$y'' = \frac{d}{dx}\left(\frac{x + 1}{y}\right)$$
$$= \frac{(y)(1) - (x + 1)y'}{y^2}$$
$$= \frac{y - (x + 1)\left(\dfrac{x + 1}{y}\right)}{y^2}$$
$$= \frac{y^2 - (x + 1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may write
$y^2 - (x + 1)^2 = (x^2 + 2x) - (x^2 + 2x + 1) = -1$, which

gives $y = -\dfrac{1}{y^3}$.

30. $y^2 + 2y = 2x + 1$

$$\frac{d}{dx}(y^2 + 2y) = \frac{d}{dx}(2x + 1)$$

$$(2y + 2)y' = 2$$

$$y' = \frac{1}{y + 1}$$

30. Continued

$$y'' = \frac{d}{dx}\frac{1}{y+1}$$
$$= -(y+1)^{-2}y'$$
$$= -(y+1)^{-2}\left(\frac{1}{y+1}\right)$$
$$= -\frac{1}{(y+1)^3}$$

31. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{9/4} = \dfrac{9}{4}x^{(9/4)-1} = \dfrac{9}{4}x^{5/4}$

32. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{-3/5} = -\dfrac{3}{5}x^{(-3/5)-1} = -\dfrac{3}{5}x^{-8/5}$

33. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[3]{x} = \dfrac{d}{dx}x^{1/3} = \dfrac{1}{3}x^{(1/3)-1} = \dfrac{1}{3}x^{-2/3}$

34. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[4]{x} = \dfrac{d}{dx}x^{1/4} = \dfrac{1}{4}x^{(1/4)-1} = \dfrac{1}{4}x^{-3/4}$

35. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x+5)^{-1/2} = -\dfrac{1}{2}(2x+5)^{(-1/2)-1}\dfrac{d}{dx}(2x+5)$

$$= -\frac{1}{2}(2x+5)^{-3/2}(2) = -(2x+5)^{-3/2}$$

36. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1-6x)^{2/3}$

$$= \frac{2}{3}(1-6x)^{(2/3)-1}\frac{d}{dx}(1-6x)$$
$$= \frac{2}{3}(1-6x)^{-1/3}(-6)$$
$$= -4(1-6x)^{-1/3}$$

37. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x\sqrt{x^2+1}\right)$

$$= x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}\frac{d}{dx}(x)$$
$$= x\frac{d}{dx}(x^2+1)^{1/2} + (x^2+1)^{1/2}$$
$$= x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x) + (x^2+1)^{1/2}$$
$$= x^2(x^2+1)^{-1/2} + (x^2+1)^{1/2}$$

Note: This answer is equivalent to $\dfrac{2x^2+1}{\sqrt{x^2+1}}$.

38. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{\sqrt{x^2+1}} = \dfrac{(x^2+1)^{1/2}\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)^{1/2}}{x^2+1}$

$$= \frac{(x^2+1)^{1/2} - x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$
$$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$$
$$= \frac{1}{(x^2+1)^{3/2}}$$
$$= (x^2+1)^{-3/2}$$

39. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1-x^{1/2})^{1/2}$

$$= \frac{1}{2}(1-x^{1/2})^{-1/2}\frac{d}{dx}(1-x^{1/2})$$
$$= \frac{1}{2}(1-x^{1/2})^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right)$$
$$= -\frac{1}{4}(1-x^{1/2})^{-1/2}x^{-1/2}$$

40. $\dfrac{dy}{dx} = \dfrac{d}{dx}3(2x^{-1/2}+1)^{-1/3}$

$$= -(2x^{-1/2}+1)^{-4/3}\frac{d}{dx}(2x^{-1/2}+1)$$
$$= -(2x^{-1/2}+1)^{-4/3}(-x^{-3/2})$$
$$= x^{-3/2}(2x^{-1/2}+1)^{-4/3}$$

41. $\dfrac{dy}{dx} = \dfrac{d}{dx}3(\csc x)^{3/2}$

$$= \frac{9}{2}(\csc x)^{1/2}\frac{d}{dx}(\csc x)$$
$$= \frac{9}{2}(\csc x)^{1/2}(-\csc x\cot x)$$
$$= -\frac{9}{2}(\csc x)^{3/2}\cot x$$

42. $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin(x+5)]^{5/4}$

$$= \frac{5}{4}[\sin(x+5)]^{1/4}\frac{d}{dx}\sin(x+5)$$
$$= \frac{5}{4}[\sin(x+5)]^{1/4}\cos(x+5)$$

43. (a) If $f(x) = \dfrac{3}{2}x^{2/3} - 3$, then

$$f'(x) = x^{-1/3} \text{ and } f''(x) = -\frac{1}{3}x^{-4/3}$$

which contradicts the given equation $f''(x) = x^{-1/3}$.

(b) If $f(x) = \dfrac{9}{10}x^{5/3} - 7$, then

$$f'(x) = \frac{3}{2}x^{2/3} \text{ and } f''(x) = x^{-1/3}, \text{ which matches the}$$

given equation.

(c) Differentiating both sides of the given equation

$$f''(x) = x^{-1/3} \text{ gives } f'''(x) = -\frac{1}{3}x^{-4/3}, \text{ so it } \textit{must} \text{ be true}$$

that $f'''(x) = -\dfrac{1}{3}x^{-4/3}$.

(d) If $f'(x) = \dfrac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$, which matches

the given equation.

Conclusion: (b), (c), and (d) could be true.

44. (a) If $g'(t) = 4\sqrt[4]{t} - 4$, then

$g''(t) = \dfrac{d}{dx}(4t^{1/4} - 4) = t^{-3/4} = \dfrac{1}{t^{3/4}}$, which matches

the given equation.

(b) Differentiating both sides of the given equation

$g''(t) = \dfrac{1}{t^{3/4}} = t^{-3/4}$ gives $g'''(t) = -\dfrac{3}{4}t^{-7/4}$, which is not

consistent with $g'''(t) = -\dfrac{4}{\sqrt[4]{t}}$.

(c) If $g(t) = t - 7 + \dfrac{16}{5}t^{5/4}$, then

$g'(t) = 1 + 4t^{1/4}$ and $g''(t) = t^{-3/4} = \dfrac{1}{t^{3/4}}$, which matches

the given equation.

(d) If $g'(t) = \dfrac{1}{4}t^{1/4}$, then $g''(t) = \dfrac{1}{16}t^{-3/4}$, which contradicts

the given equation.

Conclusion: (a) and (c) could be true.

45. (a) $y^4 = y^2 - x^2$

$\dfrac{d}{dx}(y^4) = \dfrac{d}{dx}(y^2) - \dfrac{d}{dx}x^2$

$4y^3\dfrac{dy}{dx} = 2y\dfrac{dy}{dx} - 2x$

$(4y^3 - 2y)\dfrac{dy}{dx} = -2x$

$\dfrac{dy}{dx} = \dfrac{-2x}{4y^3 - 2y} = \dfrac{x}{y - 2y^3}$

At $\left(\dfrac{\sqrt{3}}{4}, \dfrac{\sqrt{3}}{2}\right)$:

Slope $= \dfrac{\dfrac{\sqrt{3}}{4}}{\dfrac{\sqrt{3}}{2} - 2\left(\dfrac{\sqrt{3}}{2}\right)^3}$

$= \dfrac{\dfrac{\sqrt{3}}{4}}{\dfrac{\sqrt{3}}{2} - \dfrac{3\sqrt{3}}{4}} \cdot \dfrac{\dfrac{4}{\sqrt{3}}}{\dfrac{4}{\sqrt{3}}} = \dfrac{1}{2 - 3} = -1$

At $\left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{2}\right)$:

Slope $= \dfrac{\dfrac{\sqrt{3}}{4}}{\dfrac{1}{2} - 2\left(\dfrac{1}{2}\right)^3} = \dfrac{\dfrac{\sqrt{3}}{4}}{\dfrac{1}{2} - \dfrac{1}{4}} \cdot \dfrac{4}{4} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

(b)
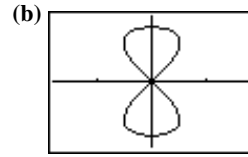
[−1.8, 1.8] by [−1.2, 1.2]
Parameter interval: $-1 \le t \le 1$

46. (a) $y^2(2 - x) = x^3$

$\dfrac{d}{dx}\left[y^2(2-x)\right] = \dfrac{d}{dx}(x^3)$

$(y^2)(-1) + (2 - x)(2y)\dfrac{dy}{dx} = 3x^2$

$2y(2 - x)\dfrac{dy}{dx} = 3x^2 + y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2 + y^2}{2y(2 - x)}$

Slope at (1, 1): $\dfrac{3(1)^2 + (1)^2}{2(1)(2 - 1)} = \dfrac{4}{2} = 2$

Tangent: $y = 2(x - 1) + 1$ or $y = 2x - 1$

Normal: $y = -\dfrac{1}{2}(x - 1) + 1$ or $y = -\dfrac{1}{2}x + \dfrac{3}{2}$

(b) One way is to graph the equations $y = \pm\sqrt{\dfrac{x^3}{2 - x}}$.

47. (a) $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal -1.

(b) $x^3y^2 = \cos(\pi y)$

$\dfrac{d}{dx}(x^3 y^2) = \dfrac{d}{dx}\cos(\pi y)$

$(x^3)(2y)\dfrac{dy}{dx} + (y^2)(3x^2) = (-\sin \pi y)(\pi)\dfrac{dy}{dx}$

$(2x^3 y + \pi \sin \pi y)\dfrac{dy}{dx} = -3x^2 y^2$

$\dfrac{dy}{dx} = -\dfrac{3x^2 y^2}{2x^3 y + \pi \sin \pi y}$

Slope at $(-1, 1): -\dfrac{3(-1)^2(1)}{2(-1)^3(1) + \pi \sin \pi} = \dfrac{-3}{-2} = \dfrac{3}{2}$

The slope of the tangent line is $\dfrac{3}{2}$.

48. (a) When $x = 2$, we have $y^3 - 2y = -1$, or $y^3 - 2y + 1 = 0$.

Clearly, $y = 1$ is one solution, and we may factor

$y^3 - 2y + 1$ as $(y - 1)(y^2 + y - 1)$. The solutions of

$y^2 + y - 1 = 0$ are $y = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2}$.

Hence, there are three possible y-values: $1, \dfrac{-1 - \sqrt{5}}{2}$,

and $\dfrac{-1 + \sqrt{5}}{2}$.

48. Continued

(b)
$$y^3 - xy = -1$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-1)$$

$$3y^2 y' - xy' - (y)(1) = 0$$

$$(3y^2 - x)y' = y$$

$$y' = \frac{y}{3y^2 - x}$$

$$y'' = \frac{d}{dx}\frac{y}{3y^2 - x}$$

$$= \frac{(3y^2 - x)(y') - (y)(6y\,y' - 1)}{(3y^2 - x)^2}$$

$$= \frac{y - xy' - 3y^2 y'}{(3y^2 - x)^2}$$

Since we are working with numerical information, there is no need to write a general expression for y'' in terms of x and y.

To evaluate $f'(2)$, evaluate the expression for y' using $x = 2$ and $y = 1$:

$$f'(2) = \frac{1}{3(1)^2 - 2} = \frac{1}{1} = 1$$

To evaluate $f''(2)$, evaluate the expression for y'' using $x = 2$, $y = 1$, and $y' = 1$:

$$f''(2) = \frac{(1) - 2(1) - 3(1)^2(1)}{[3(1)^2 - 2]^2} = \frac{-4}{1} = -4$$

49. Find the two points:

The curve crosses the x-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$.

Show tangents are parallel:

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Slope at $(\sqrt{7}, 0)$: $-\dfrac{2\sqrt{7} + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0)$: $-\dfrac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is -2.

50.
$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

(a) The tangent is parallel to the x-axis when

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = 0, \text{ or } y = -2x.$$

Substituting $-2x$ for y in the original equation, we have

$$x^2 + xy + y^2 = 7$$

$$x^2 + (x)(-2x) + (-2x)^2 = 7$$

$$x^2 - 2x^2 + 4x^2 = 7$$

$$3x^2 = 7$$

$$x = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$ and $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$.

(b) Since x and y are interchangeable in the original equation, $\dfrac{dx}{dy}$ can be obtained by interchanging x and y in the expression for $\dfrac{dy}{dx}$. That is, $\dfrac{dx}{dy} = -\dfrac{2y + x}{x + 2y}$. The tangent is parallel to the y-axis when $\dfrac{dx}{dy} = 0$, or $x = -2y$. Substituting $-2y$ for x in the original equation, we have:

$$x^2 + xy + y^2 = 7$$

$$(-2y)^2 + (-2y)(y) + y^2 = 7$$

$$4y^2 - 2y^2 + y^2 = 7$$

$$3y^2 = 7$$

$$y = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$ and $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$.

Note that these are the same points that would be obtained by interchanging x and y in the solution to part (a).

51. First curve:

$$2x^2 + 3y^2 = 5$$

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(5)$$

$$4x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x}{6y} = -\frac{2x}{3y}$$

51. Continued

Second curve:

$$y^2 = x^3$$

$$\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$

$$2y\frac{dy}{dx} = 3x^2$$
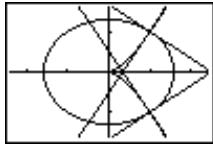
$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At (1, 1), the slopes are $-\frac{2}{3}$ and $\frac{3}{2}$ respectively.

At (1, −1), the slopes are $\frac{2}{3}$ and $-\frac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\frac{5}{2}}\cos t,\ y = \sqrt{\frac{5}{3}}\sin t$

Second curve: $x = \sqrt[3]{t^2},\ y = t$

Tangents at (1, 1): $x = 1 + 3t, y = 1 - 2t$
$$x = 1 + 2t,\ y = 1 + 3t$$

Tangents at (1, −1): $x = 1 + 3t, y = -1 + 2t$
$$x = 1 + 2t,\ y = -1 - 3t$$

[−2.4, 2.4] by [−1.6, 1.6]

52. $v(t) = s'(t) = \dfrac{d}{dt}(4+6t)^{3/2} = \dfrac{3}{2}(4+6t)^{1/2}(6)$

$$= 9(4+6t)^{1/2}$$

$$a(t) = v'(t) = \frac{d}{dt}[9(4+6t)^{1/2}]$$

$$= \frac{9}{2}(4+6t)^{-1/2}(6) = 27(4+6t)^{-1/2}$$

At $t = 2$, the velocity is $v(2) = 36$ m/sec and the acceleration is $a(2) = \dfrac{27}{4}$ m/sec^2.

53. Acceleration $= \dfrac{dv}{dt} = \dfrac{d}{dt}[8(s-t)^{1/2} + 1]$

$$= 4(s-t)^{-1/2}\left(\frac{ds}{dt} - 1\right)$$

$$= 4(s-t)^{-1/2}\,(v-1)$$

$$= 4(s-t)^{-1/2}\,[(8(s-t)^{1/2} + 1) - 1]$$

$$= 32(s-t)^{-1/2}\,(s-t)^{1/2}$$

$$= 32\text{ ft / sec}^2$$

54. $$y^4 - 4y^2 = x^4 - 9x^2$$

$$\frac{d}{dx}(y^4) - \frac{d}{dx}(4y^2) = \frac{d}{dx}(x^4) - \frac{d}{dx}(9x^2)$$

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$$

$$\frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$$

Slope at (3,2): $\dfrac{2(3)^3 - 9(3)}{2(2)^3 - 4(2)} = \dfrac{27}{8}$

Slope at (−3,2): $\dfrac{2(-3)^3 - 9(-3)}{2(2)^3 - 4(2)} = -\dfrac{27}{8}$

Slope at (−3,−2): $\dfrac{2(-3)^3 - 9(-3)}{2(-2)^3 - 4(-2)} = \dfrac{27}{8}$

Slope at (3,−2): $\dfrac{2(3)^3 - 9(3)}{2(-2)^3 - 4(-2)} = -\dfrac{27}{8}$

55. (a) $$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9(y)(1) = 0$$

$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Slope at (4, 2): $\dfrac{3(2) - (4)^2}{(2)^2 - 3(4)} = \dfrac{-10}{-8} = \dfrac{5}{4}$

Slope at (2, 4): $\dfrac{3(4) - (2)^2}{(4)^2 - 3(2)} = \dfrac{8}{10} = \dfrac{4}{5}$

(b) The tangent is horizontal when

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} = 0,\text{ or } y = \frac{x^2}{3}.$$

Substituting $\dfrac{x^2}{3}$ for y in the original equation, we have:

$$x^3 + y^3 - 9xy = 0$$

$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$

$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$

$$\frac{x^3}{27}(x^3 - 54) = 0$$

$$x = 0 \text{ or } x = \sqrt[3]{54} = 3\sqrt[3]{2}$$

At $x = 0$, we have $y = \dfrac{0^2}{3} = 0$, which gives the point (0, 0), which is the origin. At $x = 3\sqrt[3]{2}$, we have

$$y = \frac{1}{3}(3\sqrt[3]{2})^2 = \frac{1}{3}(9\sqrt[3]{4}) = 3\sqrt[3]{4},\text{ so the point other}$$

than the origin is $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ or approximately (3.780, 4.762).

55. Continued

(c) The equation $x^3 + y^3 - 9xy$ is not affected by interchanging x and y, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the x-value and the y-value in the answer to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or approximately (4.762, 3.780).

56.
$$x^2 + 2xy - 3y^2 = 0$$
$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(0)$$
$$2x + 2x\frac{dy}{dx} + 2(y)(1) - 6y\frac{dy}{dx} = 0$$
$$(2x - 6y)\frac{dy}{dx} = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y} = \frac{x + y}{3y - x}$$

At (1, 1) the curve has slope $\frac{1+1}{3(1)-1} = \frac{2}{2} = 1$, so the normal

line is $y = -1(x - 1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for y in the original equation, we have:
$$x^2 + 2xy - 3y^2 = 0$$
$$x^2 + 2x(-x + 2) - 3(-x + 2)^2 = 0$$
$$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$$
$$-4x^2 + 16x - 12 = 0$$
$$-4(x - 1)(x - 3) = 0$$
$$x = 1 \text{ or } x = 3$$

Since the given point (1,1) had $x = 1$, we choose $x = 3$ and so $y = -(3) + 2 = -1$. The desired point is $(3, -1)$.

57.
$$xy + 2x - y = 0$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$
$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$
$$(x - 1)\frac{dy}{dx} = -2 - y$$
$$\frac{dy}{dx} = \frac{-2 - y}{x - 1} = \frac{2 + y}{1 - x}$$

Since the slope of the line $2x + y = 0$ is -2, we wish to find points where the normal has slope -2, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have
$$\frac{2 + y}{1 - x} = \frac{1}{2}$$
$$2(2 + y) = 1 - x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:
$$xy + 2x - y = 0$$
$$(-2y - 3)y + 2(-2y - 3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y + 1)(y + 3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1, x = -2y - 3 = 2 - 3 = -1$.
At $y = -3: x = -2y - 3 = 6 - 3 = 3$.

The desired points are $(-1, -1)$ and $(3, -3)$.
Finally, we find the desired normals to the curve, which are the lines of slope -2 passing through each of these points.
At $(-1, -1)$, the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At $(3, -3)$, the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

58.
$$x = y^2$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$
$$1 = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at (x, y) has slope $-2y$. Thus, the normal line at (b^2, b) is $y = -2b(x - b^2) + b$, or $y = -2bx + 2b^3 + b$.

This line intersects the x-axis at $x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$,

which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes ± 1, which gives

$-2y = \pm 1$ or $y = \pm\frac{1}{2}\left(\text{or } b = \pm\frac{1}{2}\right)$. The corresponding value

of a is $b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal

normals are perpendicular when $a = \frac{3}{4}$.

59. False.
$$\frac{d}{dx}(xy^2 + x) = \frac{d}{dx}(1)$$
$$y^2 + 1 + 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-1 - y}{2xy}, \frac{-1 - 1}{2\left(\frac{1}{2}\right)1} = -2$$

60. True. By the power rule.
$$y = (x)^{1/3}$$
$$\frac{dy}{dx} = \frac{d}{dx}(x)^{1/3} = \frac{1}{3}x^{2/3} = \frac{1}{3x^{2/3}}$$

61. A. $\dfrac{d}{dx}(x^2 - xy + y^2) = \dfrac{d}{dx}(1)$

$2x - y + (-x + 2y)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$

62. A. $\dfrac{d^2y}{dx} = \dfrac{d}{dx}\left(\dfrac{y - 2x}{2y - x}\right) = \dfrac{d}{dx}(0)$

$= \dfrac{-3y + 3x}{(x - 2y)^2}(y')$

$= -\dfrac{6}{(2y - x)^3}$

63. E. $\dfrac{d}{dx}(y) = \dfrac{d}{dx}x^{3/4}$

$\dfrac{dy}{dx} = \dfrac{3}{4}x^{-1/4} = \dfrac{3}{4x^{1/4}}$

64. C. $\dfrac{d}{dx}(y^2 - x^2) = \dfrac{d}{dx}(1)$

$-2x + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{1}{\sqrt{2}}$

65. (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$b^2x^2 + a^2y^2 = a^2b^2$

$\dfrac{d}{dx}(b^2x^2) + \dfrac{d}{dx}(a^2y^2) = \dfrac{d}{dx}(a^2b^2)$

$2b^2x + 2a^2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2b^2x}{2a^2y} = \dfrac{b^2x}{a^2y}$

The slope at (x_1, y_1) is $-\dfrac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\dfrac{b^2x_1}{a^2y_1}(x - x_1)$. This gives:

$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$

$a^2y_1y + b^2x_1x = a^2y_1^2 + b^2x_1^2$.

But $a^2y_1^2 + b^2x_1^2 = a^2b^2$ since (x_1, y_1) is on the ellipse.

Therefore, $a^2y_1y + b^2x_1x = a^2b^2$, and dividing by

a^2b^2 gives $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$.

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$b^2x^2 - a^2y^2 = a^2b^2$

$\dfrac{d}{dx}(b^2x^2) - \dfrac{d}{dx}(a^2y^2) = \dfrac{d}{dx}(a^2b^2)$

$2b^2x - 2a^2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-2b^2x}{-2a^2y} = \dfrac{b^2x}{a^2y}$

The slope at (x_1, y_1) is $\dfrac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\dfrac{b^2x_1}{a^2y_1}(x - x_1)$.

This gives:

$a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$

$b^2x_1^2 - a^2y_1^2 = b^2x_1x - a^2y_1y$

But $b^2x_1^2 - a^2y_1^2 = a^2b^2$ since (x_1, y_1) is on the

hyperbola. Therefore, $b^2x_1x - a^2y_1y = a^2b^2$, and

dividing by a^2b^2 gives $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$.

66. (a) Solve for y:

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$-\dfrac{y^2}{b^2} = -\dfrac{x^2}{a^2} + 1$

$y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$

$y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$

(b) $\lim\limits_{x \to \infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty}\dfrac{\dfrac{b}{a}\sqrt{x^2 - a^2}}{\dfrac{b}{a}|x|}$

$= \lim\limits_{x \to \infty}\dfrac{\sqrt{x^2 - a^2}}{\sqrt{x^2}}$

$= \lim\limits_{x \to \infty}\sqrt{1 - \dfrac{a^2}{x^2}} = 1$

(c) $\lim\limits_{x \to \infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty}\dfrac{-\dfrac{b}{a}\sqrt{x^2 - a^2}}{-\dfrac{b}{a}|x|}$
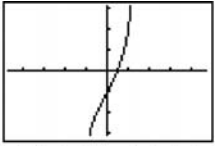
$= \lim\limits_{x \to \infty}\dfrac{\sqrt{x^2 - a^2}}{\sqrt{x^2}}$

$= \lim\limits_{x \to \infty}\sqrt{1 - \dfrac{a^2}{x^2}} = 1$

Section 3.8 Derivatives of Inverse Trigonometric Functions (pp. 165–171)

Exploration 1 Finding a derivative on an Inverse Graph Geometrically

1. The graph is shown at the right. It appears to be a one-to-one function

[−4.7, 4.7] by [−3.1, 3.1]

2. $f'(x) = 5x^4 + 2$. The fact that this function is always positive enables us to conclude that f is everywhere increasing, and hence one-to-one.

3. The graph of f^{-1} is shown to the right, along with the graph of f. The graph of f^{-1} is obtained from the graph of f by reflecting it in the line $y = x$.
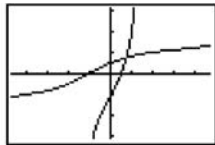
[−4.7, 4.7] by [−3.1, 3.1]

4. The line L is tangent to the graph of f^{-1} at the point $(2, 1)$.
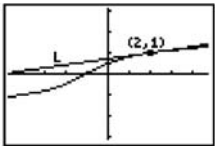
[−4.7, 4.7] by [−3.1, 3.1]

5. The reflection of line L is tangent to the graph of f at the point
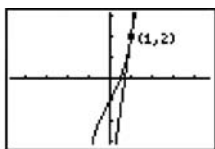
[−4.7, 4.7] by [−3.1, 3.1]

6. The reflection of the line L is the tangent line to the graph of $y = x^5 + 2x - 1$ at the point $(1, 2)$. The slope is $\dfrac{dy}{dx}$ at $x = 1$, which is 7.

7. The slope of L is the reciprocal of the slope of its reflection $\left(\text{since } \dfrac{\Delta y}{\Delta x} \text{ gets reflected to become } \dfrac{\Delta x}{\Delta y} \right)$. It is $\dfrac{1}{7}$.

8. $\dfrac{1}{7}$

Quick Review 3.8

1. Domain: $[-1, 1]$

   Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$

   At 1: $\dfrac{\pi}{2}$

2. Domain: $[-1, 1]$

   Range: $\left[0, \pi \right]$

   At 1: 0

3. Domain: all reals

   Range: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right)$

   At 1: $\dfrac{\pi}{4}$

4. Domain: $(-\infty, -1] \cup [1, \infty)$

   Range: $\left[0, \dfrac{\pi}{2} \right) \cup \left(\dfrac{\pi}{2}, \pi \right]$

   At 1: 0

5. Domain: all reals
   Range: all reals
   At 1:1

6. $f(x) = y = 3x - 8$
   $y + 8 = 3x$
   $x = \dfrac{y + 8}{3}$
   Interchange x and y:
   $y = \dfrac{x + 8}{3}$
   $f^{-1}(x) = \dfrac{x + 8}{3}$

7. $f(x) = y = \sqrt[3]{x + 5}$
   $y^3 = x + 5$
   $x = y^3 - 5$
   Interchange x and y:
   $y = x^3 - 5$
   $f^{-1}(x) = x^3 - 5$

8. $f(x) = y = \dfrac{8}{x}$
   $x = \dfrac{8}{y}$
   Interchange x and y:
   $y = \dfrac{8}{x}$
   $f^{-1}(x) = \dfrac{8}{x}$

9. $f(x) = y = \dfrac{3x-2}{x}$

$xy = 3x - 2$

$(y-3)x = -2$

$x = \dfrac{-2}{y-3} = \dfrac{2}{3-y}$

Interchange x and y:

$y = \dfrac{2}{3-x}$

$f^{-1}(x) = \dfrac{2}{3-x}$

10. $f(x) = y = \arctan\dfrac{x}{3}$

$\tan y = \dfrac{x}{3}, \ -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$x = 3\tan y, \ -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Interchange x and y:

$y = 3\tan x, \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$f^{-1}(x) = 3\tan x, \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

Section 3.8 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos^{-1}(x^2) = -\dfrac{1}{\sqrt{1-(x^2)^2}}\dfrac{d}{dx}(x^2)$

$= -\dfrac{1}{\sqrt{1-x^4}}(2x) = -\dfrac{2x}{\sqrt{1-x^4}}$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos^{-1}\left(\dfrac{1}{x}\right) = -\dfrac{1}{\sqrt{1-\left(\dfrac{1}{x}\right)^2}}\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

$= -\dfrac{1}{\sqrt{1-\left(\dfrac{1}{x}\right)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$

3. $\dfrac{dy}{dt} = \dfrac{d}{dt}\sin^{-1}\sqrt{2t} = \dfrac{1}{\sqrt{1-(\sqrt{2t})^2}}\dfrac{d}{dt}(\sqrt{2t}) = \dfrac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $\dfrac{dy}{dt} = \dfrac{d}{dt}\sin^{-1}(1-t) = \dfrac{1}{\sqrt{1-(1-t)^2}}\dfrac{d}{dt}(1-t)$

$= -\dfrac{1}{\sqrt{2t-t^2}}$

5. $\dfrac{dy}{dt} = \dfrac{d}{dt}\sin^{-1}\left(\dfrac{3}{t^2}\right) = \dfrac{1}{\sqrt{1-\left(\dfrac{3}{t^2}\right)^2}}\dfrac{d}{dt}\left(\dfrac{3}{t^2}\right)$

$= \dfrac{1}{\sqrt{1-\dfrac{9}{t^4}}}\left(-\dfrac{6}{t^3}\right) = -\dfrac{6}{t\sqrt{t^4-9}}$

6. $\dfrac{dy}{ds} = \dfrac{d}{ds}(s\sqrt{1-s^2}) + \dfrac{d}{ds}(\cos^{-1}s)$

$= (s)\left(\dfrac{1}{2\sqrt{1-s^2}}\right)(-2s) + (\sqrt{1-s^2})(1) - \dfrac{1}{\sqrt{1-s^2}}$

$= -\dfrac{s^2}{\sqrt{1-s^2}} + \sqrt{1-s^2} - \dfrac{1}{\sqrt{1-s^2}}$

$= -\dfrac{-s^2 + (1-s^2) - 1}{\sqrt{1-s^2}}$

$= -\dfrac{2s^2}{\sqrt{1-s^2}}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sin^{-1}x) + \dfrac{d}{dx}(\sqrt{1-x^2})$

$= (x)\left(\dfrac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1}x)(1) + \dfrac{1}{2\sqrt{1-x^2}}(-2x)$

$= \sin^{-1}x$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin^{-1}(2x)]^{-1}$

$= -[\sin^{-1}(2x)]^{-2}\dfrac{d}{dx}\sin^{-1}(2x)$

$= -[\sin^{-1}(2x)]^{-2}\dfrac{1}{\sqrt{1-4x^2}}(2)$

$= -\dfrac{2}{[\sin^{-1}(2x)]^2\sqrt{1-4x^2}}$

9. $x(t) = \sin^{-1}\left(\dfrac{t}{4}\right)$

$y = \sin^{-1}u \qquad u = \dfrac{t}{4}$

$\sin y = u \qquad \dfrac{du}{dt} = \dfrac{1}{4}$

$\dfrac{d}{du}(\sin y) = \dfrac{d}{du}u$

$\cos y\dfrac{dy}{du} = 1$

$\dfrac{dy}{du} = \dfrac{1}{\cos y}$

$\dfrac{d}{du}(\sin^{-1}u) = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

$v(t) = \dfrac{1}{4\sqrt{1-t^2/16}}$

$v(3) = \dfrac{1}{4\sqrt{1-9/16}} = \dfrac{\sqrt{7}}{7}$

10. See Problem 9.

$u = \dfrac{\sqrt{t}}{4}$

$\dfrac{du}{dt} = \dfrac{1}{8\sqrt{t}}$

$v(t) = \dfrac{1}{8\sqrt{t}\sqrt{1-t/16}}$

$v(4) = \dfrac{1}{8\sqrt{4}\sqrt{1-4/16}} = \dfrac{\sqrt{3}}{24}$

11. $x(t) = \tan^{-1} t$

$y = \tan^{-1} t$

$\dfrac{d}{dt} \tan y = \dfrac{d}{dt} t$

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1 + (\tan y)^2}$

$\qquad = \dfrac{1}{1 + t^2}$

$x(2) = \dfrac{1}{1 + 2^2}$

$\qquad = \dfrac{1}{5}$

12. See problem 11.

$x(t) = \dfrac{2t}{1 + t^2}$

$x(1) = \dfrac{2(1)}{1 + (1)^2} = 1$

13. $\dfrac{dy}{ds} = \dfrac{d}{ds} \sec^{-1}(2s + 1)$

$= \dfrac{1}{|2s+1|\sqrt{(2s+1)^2 - 1}} \dfrac{d}{ds}(2s+1)$

$= \dfrac{1}{|2s+1|\sqrt{4s^2 + 4s}}(2) = \dfrac{1}{|2s+1|\sqrt{s^2 + s}}$

14. $\dfrac{dy}{ds} = \dfrac{d}{ds} \sec^{-1} 5s = \dfrac{1}{|5s|\sqrt{(5s)^2 - 1}} \dfrac{d}{ds}(5s) = \dfrac{1}{|s|\sqrt{25s^2 - 1}}$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx} \csc^{-1}(x^2 + 1)$

$= -\dfrac{1}{|x^2+1|\sqrt{(x^2+1)^2 - 1}} \dfrac{d}{dx}(x^2 + 1)$

$= -\dfrac{2x}{(x^2+1)\sqrt{x^4 + 2x^2}} = -\dfrac{2}{(x^2+1)\sqrt{x^2 + 2}}$

Note that the condition $x > 0$ is required in the last step.

16. $\dfrac{dy}{dx} = \dfrac{d}{dx} \csc^{-1}\left(\dfrac{x}{2}\right) = -\dfrac{1}{\left|\dfrac{x}{2}\right|\sqrt{\left(\dfrac{x}{2}\right)^2 - 1}} \dfrac{d}{dx}\left(\dfrac{x}{2}\right)$

$= -\dfrac{2}{|x|\sqrt{x^2 - 4}}$

17. $\dfrac{dy}{dt} = \dfrac{d}{dt} \sec^{-1}\left(\dfrac{1}{t}\right) = \dfrac{1}{\left|\dfrac{1}{t}\right|\sqrt{\left(\dfrac{1}{t}\right)^2 - 1}} \dfrac{d}{dt}\left(\dfrac{1}{t}\right)$

$= \dfrac{1}{\left|\dfrac{1}{t}\right|\sqrt{\left(\dfrac{1}{t}\right)^2 - 1}}\left(-\dfrac{1}{t^2}\right) = -\dfrac{1}{\sqrt{1 - t^2}}$

Note that the condition $t > 0$ is required in the last step.

18. $\dfrac{dy}{dt} = \dfrac{d}{dt} \cot^{-1} \sqrt{t} = -\dfrac{1}{1 + (\sqrt{t})^2} \dfrac{d}{dt} \sqrt{t}$

$= -\dfrac{1}{2\sqrt{t}(t+1)}$

19. $\dfrac{dy}{dt} = -\dfrac{d}{dt} \cot^{-1} \sqrt{t-1} = -\dfrac{1}{1 + (\sqrt{t-1})^2} \dfrac{d}{dt} \sqrt{t-1}$

$= -\left(\dfrac{1}{1 + t - 1}\right)\left(\dfrac{1}{2\sqrt{t-1}}\right) = -\dfrac{1}{2t\sqrt{t-1}}$

20. $\dfrac{dy}{ds} = \dfrac{d}{ds} \sqrt{s^2 - 1} - \dfrac{d}{ds} \sec^{-1} s$

$= \dfrac{1}{2\sqrt{s^2 - 1}}(2s) - \dfrac{1}{|s|\sqrt{s^2 - 1}}$

$= \dfrac{s|s| - 1}{|s|\sqrt{s^2 - 1}}$

21. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\tan^{-1} \sqrt{x^2 - 1}) + \dfrac{d}{dx}(\csc^{-1} x)$

$= \dfrac{1}{1 + (\sqrt{x^2 - 1})^2} \dfrac{d}{dx}(\sqrt{x^2 - 1}) - \dfrac{1}{|x|\sqrt{x^2 - 1}}$

$= \dfrac{1}{x^2} \dfrac{1}{2\sqrt{x^2 - 1}}(2x) - \dfrac{1}{|x|\sqrt{x^2 - 1}}$

$= \dfrac{1}{x\sqrt{x^2 - 1}} - \dfrac{1}{|x|\sqrt{x^2 - 1}}$

$= 0$

Note that the condition $x > 1$ is required in the last step.

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cot^{-1} \dfrac{1}{x}\right) - \dfrac{d}{dx}(\tan^{-1} x)$

$= -\dfrac{1}{1 + \left(\dfrac{1}{x^2}\right)} \dfrac{d}{dx}\left(\dfrac{1}{x}\right) - \dfrac{1}{1 + x^2}$

$= \left(-\dfrac{1}{1 + \dfrac{1}{x^2}}\right)\left(-\dfrac{1}{x^2}\right) - \dfrac{1}{1 + x^2}$

$= \dfrac{1}{x^2 + 1} - \dfrac{1}{1 + x^2}$

$= 0, \; x \neq 0$

The condition $x \neq 0$ is required because the original function was undefined when $x \neq 0$.

23.
$$y = \sec^{-1} x$$
$$\frac{dy}{dx}\sec y = \frac{d}{dx}x$$
$$\sec y \ \tan y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec y \ \tan y}$$
$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$
$$y'(2) = \frac{1}{2\sqrt{4-1}} = \frac{1}{2\sqrt{3}} = 0.289$$
$$y(2) = \sec^{-1} 2 = 2.203$$
$$y = 0.289(x - 2) + 2.203$$
$$y = 0.289x + 1.625$$

24.
$$y = \tan^{-1}(x)$$
$$\frac{d}{dx}(\tan y) = \frac{d}{dx}x$$
$$\sec^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$= \frac{1}{1 + (\tan y)^2}$$
$$= \frac{1}{1 + x^2}$$
$$y'(2) = \frac{1}{1 + 2^2}$$
$$y'(2) = \frac{1}{5}$$
$$y(2) = \tan^{-1}(x) = 1.107$$
$$y = \frac{1}{5}(x - 2) + 1.107$$
$$y = \frac{1}{5}x + 0.707$$

25. $y = \sin^{-1}\left(\dfrac{x}{4}\right)$

$$y = \sin^{-1} u \qquad u = \frac{x}{4}$$
$$\sin y = u \qquad \frac{du}{dx} = \frac{1}{4}$$
$$\frac{d}{du}(\sin y) = \frac{d}{d}u$$
$$\cos y \frac{dy}{du} = 1$$
$$\frac{dy}{du} = \frac{1}{\cos y}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}$$
$$y' = \frac{1}{4\sqrt{1 - \dfrac{x^2}{16}}}$$
$$y'(3) = \frac{1}{4\sqrt{1 - \dfrac{9}{16}}}$$
$$y'(3) = 0.378$$
$$y(3) = \sin^{-1}\left(\frac{3}{4}\right) = 0.848$$
$$y = 0.378(x - 3) + 0.848$$
$$y = 0.378x - 0.286$$

26. See problem 24.
$$u = x^2$$
$$\frac{du}{dx} = 2x$$
$$\frac{1}{1 + u^2}\frac{du}{dx}$$
$$\frac{2x}{1 + x^2}$$
$$= \frac{2(1)}{1 + (1)^2}$$
$$= 1$$
$$y(1) = \tan^{-1}(1)^2$$
$$y(1) = 0.785$$
$$y = 1(x - 1) + 0.785$$
$$y = x - 0.215$$

27. (a) Since $\dfrac{dy}{dx} = \sec^2 x$, the slope at $\left(\dfrac{\pi}{4}, 1\right)$ is $\sec^2\left(\dfrac{\pi}{4}\right) = 2$.

The tangent line is given by

$$y = 2\left(x - \frac{\pi}{4}\right) + 1, \text{ or } y = 2x = \frac{\pi}{2} + 1.$$

(b) Since $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, the slope at $\left(1, \dfrac{\pi}{4}\right)$ is $\dfrac{1}{1 + 1^2} = \dfrac{1}{2}$.

The tangent line is given by $x \neq 0$.

28. (a) Note that $f'(x) = 5x^4 + 6x^2 + 1$. Thus $f(1) = 3$ and
$$f'(1) = 12.$$

(b) Since the graph of $y = f(x)$ includes the point $(1, 3)$ and the slope of the graph is 12 at this point, the graph of
$$y = f^{-1}(x) \text{ will include } (3, 1) \text{ and the slope will be } \frac{1}{12}.$$

Thus, $f^{-1}(3) = 1$ and $(f^{-1})'(3) = \dfrac{1}{12}$. (We have assumed

that $f^{-1}(x)$ is defined and differentiable at
$x = 3$. This is true by Theorem 3, because
$f'(x) = 5x^2 + 6x^2 + 1$, which is never zero.)

29. (a) Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus f is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so f has a differentiable inverse by Theorem 3.

(b) $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

(c) Since the graph of $y = f(x)$ includes the point $(0, 1)$ and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include $(1, 0)$ and the slope will be $\dfrac{1}{3}$,

Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \dfrac{1}{3}$.

30.
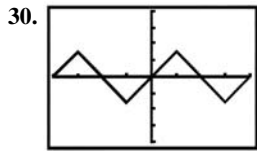
$[-2\pi, 2\pi]$ by $[-4, 4]$

(a) All reals

(b) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(c) At the points $x = k\dfrac{\pi}{2}$, where k is an odd integer.

(d)
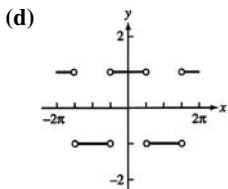
(e) $f'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x)$

$= \dfrac{1}{\sqrt{1 - \sin^2 x}}\dfrac{d}{dx}\sin x$

$= \dfrac{\cos x}{\sqrt{1 - \sin^2 x}}$

which is ± 1 depending on whether $\cos x$ is positive or negative.

31. (a) $v(t) = \dfrac{dx}{dt} = \dfrac{1}{1 + t^2}$ which is always positive.

(b) $a(t) = \dfrac{dv}{dt} = -\dfrac{2t}{(1 + t^2)^2}$ which is always negative.

(c) $\dfrac{\pi}{2}$

32. $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \sin^{-1}x\right)$

$= 0 - \dfrac{d}{dx}\sin^{-1}(x)$

$= -\dfrac{1}{\sqrt{1 - x^2}}$

33. $\dfrac{d}{dx}\cot^{-1}x = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \tan^{-1}(x)\right)$

$= 0 - \dfrac{d}{dx}\tan^{-1}(x)$

$= -\dfrac{1}{1 + x^2}$

34. $\dfrac{d}{dx}\csc^{-1}(x) = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \sec^{-1}(x)\right)$

$= 0 - \dfrac{d}{dx}\sec^{-1}(x)$

$= -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

35. True. By definition of the Function.

36. False. The domain is all real numbers.

37. E. $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{2}\right)$

$y = \sin^{-1}x \qquad u = \dfrac{x}{2}$

$\sin y = x \qquad \dfrac{du}{dx} = \dfrac{1}{2}$

$\dfrac{d}{dx}\sin y = \dfrac{d}{dx}x$

$\cos y \dfrac{dy}{dx} = 1$

$\dfrac{d}{dx} = \dfrac{1}{\cos y}$

$\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}$

$\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1 - \dfrac{x}{2}}}\dfrac{1}{2}$

$= \dfrac{1}{\sqrt{4 - x^2}}$

38. D. $\dfrac{d}{dx}\tan^{-1}(3x)$

$y = \tan^{-1}u \qquad u = 3x$

$\dfrac{d}{du}\tan y = \dfrac{d}{du}u \qquad \dfrac{du}{dx} = 3$

$\sec^2 y \dfrac{dy}{du} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

$= \dfrac{1}{1 + (\tan y)^2}$

$= \dfrac{1}{1 + u^2}\dfrac{du}{dx}$

$= \dfrac{3}{1 + 9x^2}$

39. A. $\dfrac{d}{dx}\sec^{-1}(x^2)$

$$y = \sec^{-1}u \qquad\qquad u = x^2$$

$$\frac{d}{du}(\sec y) = \frac{d}{du}u \qquad du = 2x$$

$$\sec y\tan y\frac{dy}{du} = 1$$

$$\frac{dy}{du} = \frac{1}{\sec y\tan y}$$

$$\frac{dy}{du} = \frac{1}{4\sqrt{u^2-1}}\frac{du}{dx}$$

$$= \frac{2x}{x^2\sqrt{x^4-1}}$$

$$= \frac{2}{x\sqrt{x^4-1}}$$

40. C. $\dfrac{d}{dx}\tan^{-1}(2x)$

See problem 38.

$$u = 2x$$

$$\frac{du}{dx} = 2$$

$$\frac{1}{1+u^2}\frac{du}{dx} = \frac{2}{1+4x^2}$$

$$\frac{2}{1+4(1)^2} = \frac{2}{5}$$

41. (a) $y = \dfrac{\pi}{2}$

(b) $y = -\dfrac{\pi}{2}$

(c) None, since $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}\neq 0.$

42. (a) $y = 0$

(b) $y = \pi$

(c) None, since $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{1}{1+x^2}\neq 0.$

43. (a) $y = \dfrac{\pi}{2}$

(b) $y = \dfrac{\pi}{2}$

(c) None, since $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}\neq 0.$

44. (a) $y = 0$

(b) $y = 0$

(c) None, since $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{1}{|x|\sqrt{x^2-1}}\neq 0.$

45. (a) None, since $\sin^{-1}x$ is undefined for $x > 1$.

(b) None, since $\sin^{-1}x$ is undefined for $x < -1$.

(c) None, since $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}\neq 0.$

46. (a) None, since $\cos^{-1}x$ is undefined for $x > 1$.

(b) None, since $\cos^{-1}x$ is undefined for $x < -1$.

(c) None, since $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}}\neq 0.$

47. (a)
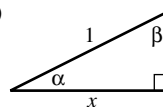
$\alpha = \cos^{-1}x,\ \beta = \sin^{-1}x$

So $\cos^{-1}x + \sin^{-1}x = \alpha + \beta = \dfrac{\pi}{2}.$

(b)
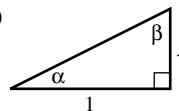
$\alpha = \tan^{-1}x,\ \beta = \cot^{-1}x$

So $\tan^{-1}x + \cot^{-1}x = \alpha + \beta = \dfrac{\pi}{2}.$

(c)
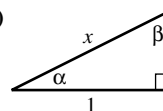
$\alpha = \sec^{-1}x,\ \beta = \csc^{-1}x$

So $\sec^{-1}x + \csc^{-1}x = \alpha + \beta = \dfrac{\pi}{2}.$

48.
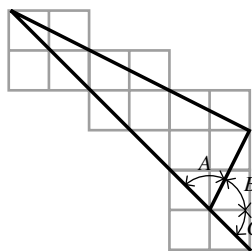
The "straight angle" with the arrows in it is the sum of the three angles A, B, and C.

A is equal to $\tan^{-1}3$ since the opposite side is 3 times as long as the adjacent side.

B is equal to $\tan^{-1}2$ since the side opposite it is 2 units and the adjacent side is one unit.

C is equal to $\tan^{-1}1$ since both the opposite and adjacent sides are one unit long.
But the sum of these three angles is the "straight angle," which has measure π radians.

49.
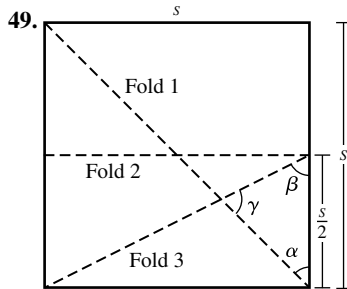
If s is the length of a side of the square, then

$\tan \alpha = \dfrac{s}{s} = 1$, so $\alpha = \tan^{-1} 1$ and

$\tan \beta = \dfrac{s}{\frac{s}{2}} = 2$, so $\beta = \tan^{-1} 2$.

From Exercise 34, we have

$\gamma = \pi - \alpha - \beta = \pi - \tan^{-1} 1 - \tan^{-1} 2 = \tan^{-1} 3$.

Section 3.9 Derivatives of Exponential and Logarithmic Functions (pp. 172–180)

Exploration 1   Leaving Milk on the Counter

1. The temperature of the refrigerator is $42°F$, the temperature of the milk at time $t = 0$.

2. The temperature of the room is $72°F$, the limit to which y tends as t increases.

3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2 (0.98)^t$ is negative, so y' (the rate at which the milk is warming) is maximized at the lowest value of t.

4. We set $y = 55$ and solve;

$72 - 30(0.98)^t = 55$

$(0.98)^t = \dfrac{17}{30}$

$t \ln(0.98) = \ln\left(\dfrac{17}{30}\right)$

$t = \dfrac{\ln\left(\dfrac{17}{30}\right)}{\ln(0.98)} = 28.114$

The milk reaches a temperature of $55°$ F after about 28 minutes.

5. $\dfrac{dy}{dx} = -30 \ln(0.98) \cdot (0.98)^t$. At $t = \dfrac{\ln\left(\dfrac{17}{30}\right)}{\ln(0.98)}$,

$\dfrac{dy}{dx} \approx 0.343$ degrees/minute.

Quick Review 3.9

1. $\log_5 8 = \dfrac{\ln 8}{\ln 5}$

2. $7^x = e^{\ln 7^x} = e^{x \ln 7}$

3. $\ln (e^{\tan x}) = \tan x$

4. $\ln (x^2 - 4) - \ln (x + 2) = \ln \dfrac{x^2 - 4}{x + 2}$

$= \ln \dfrac{(x + 2)(x - 2)}{x + 2} = \ln (x - 2)$

5. $\log_2 (8^{x-5}) = \log_2 (2^3)^{x-5} = \log_2 2^{3x-15} = 3x - 15$

6. $\dfrac{\log_4 x^{15}}{\log_4 x^{12}} = \dfrac{15 \log_4 x}{12 \log_4 x} = \dfrac{15}{12} = \dfrac{5}{4}, x > 0$

7. $3 \ln x - \ln 3x + \ln(12x^2) = \ln x^3 - \ln 3x + \ln(12x^2)$

$= \ln \dfrac{(x^3)(12x^2)}{3x} = \ln (4x^4)$

8. $3^x = 19$

$\ln 3^x = \ln 19$

$x \ln 3 = \ln 19$

$x = \dfrac{\ln 19}{\ln 3} \approx 2.68$

9. $5^t \ln 5 = 18$

$5^t = \dfrac{18}{\ln 5}$

$\ln 5^t = \ln \dfrac{18}{\ln 5}$

$t \ln 5 = \ln 18 - \ln (\ln 5)$

$t = \dfrac{\ln 18 - \ln (\ln 5)}{\ln 5} \approx 1.50$

10. $3^{x+1} = 2x$

$\ln 3^{x+1} = \ln 2^x$

$(x + 1) \ln 3 = x \ln 2$

$x(\ln 3 - \ln 2) = -\ln 3$

$x = \dfrac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$

Section 3.9 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2e^x) = 2e^x$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{2x}) = e^{2x} \dfrac{d}{dx}(2x) = 2e^{2x}$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x} = e^{-x} \dfrac{d}{dx}(-x) = -e^{-x}$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-5x} = e^{-5x} \dfrac{d}{dx}(-5x) = -5e^{-5x}$

5. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{2x/3} = e^{2x/3} \dfrac{d}{dx}\left(\dfrac{2x}{3}\right) = \dfrac{2}{3}e^{2x/3}$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{-x/4} = e^{-x/4} \dfrac{d}{dx}\left(-\dfrac{x}{4}\right) = -\dfrac{1}{4}e^{-x/4}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(xe^2) - \dfrac{d}{dx}(e^x) = e^2 - e^x$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 e^x) - \dfrac{d}{dx}(xe^x)$

$\quad = (x^2)(e^x) + (e^x)(2x) - [(x)(e^x) + (e^x)(1)]$

$\quad = x^2 e^x + xe^x - e^x$

9. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\dfrac{d}{dx}(\sqrt{x}) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{(x^2)} = e^{(x^2)}\dfrac{d}{dx}(x^2) = 2xe^{(x^2)}$

11. $\dfrac{dy}{dx} = \dfrac{d}{dx}8^x = 8^x \ln 8$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}9^{-x} = 9^{-x}(\ln 9)\dfrac{d}{dx}(-x) = -9^{-x}\ln 9$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\csc x} = 3^{\csc x}(\ln 3)\dfrac{d}{dx}(\csc x)$

$\quad\quad = 3^{\csc x}(\ln 3)(-\csc x \cot x)$

$\quad\quad = -3^{\csc x}(\ln 3)(\csc x \cot x)$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}3^{\cot x} = 3^{\cot x}(\ln 3)\dfrac{d}{dx}(\cot x)$

$\quad\quad = 3^{\cot x}(\ln 3)(-\csc^2 x)$

$\quad\quad = -3^{\cot x}(\ln 3)(\csc^2 x)$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^2) = \dfrac{1}{x^2}\dfrac{d}{dx}(x^2) = \dfrac{1}{x^2}(2x) = \dfrac{2}{x}$

16. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln x)^2 = 2\ln x\dfrac{d}{dx}(\ln x) = \dfrac{2\ln x}{x}$

17. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(x^{-1}) = \dfrac{d}{dx}(-\ln x) = -\dfrac{1}{x},\ x > 0$

18. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln\dfrac{10}{x} = \dfrac{d}{dx}(\ln 10 - \ln x) = 0 - \dfrac{1}{x}$

$\quad\quad = -\dfrac{1}{x},\ x > 0$

19. $\dfrac{d}{dx}\ln(\ln x) = \dfrac{1}{\ln x}\dfrac{d}{dx}\ln x = \dfrac{1}{\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x\ln x}$

20. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\ln x - x) = (x)\left(\dfrac{1}{x}\right) + (\ln x)(1) - 1$

$\quad\quad = 1 + \ln x - 1 = \ln x$

21. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_4 x^2) = \dfrac{d}{dx}\dfrac{\ln x^2}{\ln 4} = \dfrac{d}{dx}\left[\left(\dfrac{2}{\ln 4}\right)(\ln x)\right]$

$\quad = \dfrac{2}{\ln 4}\cdot\dfrac{1}{x} = \dfrac{2}{x\ln 4} = \dfrac{1}{x\ln 2}$

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_5 \sqrt{x}) = \dfrac{d}{dx}\dfrac{\ln x^{1/2}}{\ln 5} = \dfrac{d}{dx}\dfrac{\frac{1}{2}\ln x}{\ln 5}$

$\quad = \dfrac{1}{2\ln 5}\dfrac{d}{dx}(\ln x) = \dfrac{1}{2\ln 5}\cdot\dfrac{1}{x} = \dfrac{1}{2x\ln 5},\ x > 0$

23. $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_2\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(-\log_2 x) = -\dfrac{1}{x\ln 2},\ x > 0$

24. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{1}{\log_2 x} = -\dfrac{1}{(\log_2 x)^2}\dfrac{d}{dx}(\log_2 x)$

$\quad = -\dfrac{1}{(\log_2 x)^2}\dfrac{1}{x\ln 2} = -\dfrac{1}{x(\ln 2)(\log_2 x)^2}$

$\quad\text{or } -\dfrac{\ln 2}{x(\ln x)^2}$

25. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln 2 \cdot \log_2 x) = (\ln 2)\dfrac{d}{dx}(\log_2 x)$

$\quad = (\ln 2)\left(\dfrac{1}{x\ln 2}\right) = \dfrac{1}{x},\ x > 0$

26. $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_3(1 + x\ln 3)$

$\quad = \dfrac{1}{(1 + x\ln 3)\ln 3}\dfrac{d}{dx}(1 + x\ln 3)$

$\quad = \dfrac{\ln 3}{(1 + x\ln 3)\ln 3} = \dfrac{1}{1 + x\ln 3},\ x > -\dfrac{1}{\ln 3}$

27. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\log_{10} e^x) = \dfrac{d}{dx}(x\log_{10} e) = \log_{10} e = \dfrac{\ln e}{\ln 10}$

$\quad = \dfrac{1}{\ln 10}$

28. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln 10^x = \dfrac{d}{dx}(x\ln 10) = \ln 10$

29. $m = 5$

$\quad y = 3^x + 1$

$\quad y' = 3^x \ln 3 = 5$

$\quad x = 1.379$

$\quad y = 3^{1.379} + 1 = 5.551$

$\quad (1.379,\ 5.551)$

30. $m_2 = -\dfrac{1}{m_1} = \dfrac{1}{3}$

$\quad \dfrac{d}{dx}(2e^{x-1}) = 2e^x$

$\quad \dfrac{1}{3} = 2e^x$

$\quad \dfrac{1}{6} = e^x$

$\quad x = -1.792$

$\quad y = 2e^{x-1}$

$\quad y = \dfrac{2}{6} - 1 = -0.667$

$\quad (-1.792,\ -0.667)$

31. $y = \ln 2x$

$\quad e^y = 2x$

$\quad \dfrac{d}{dx}(e^y) = \dfrac{d}{dx}(2x)$

$\quad e^y\dfrac{dy}{dx} = 2$

$\quad \dfrac{dy}{dx} = 2e^{-1}$

32. $y = \ln(x/3)$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$e^y \frac{dy}{dx} = \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}e^{-1}$$

33. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^\pi) = \pi x^{\pi-1}$

34. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{1+\sqrt{2}}) = (1+\sqrt{2})x^{1+\sqrt{2}-1} = (1+\sqrt{2})x^{\sqrt{2}}$

35. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{-\sqrt{2}} = -\sqrt{2}x^{-\sqrt{2}-1}$

36. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{1-e} = (1-e)x^{1-e-1} = (1-e)x^{-e}$

37. $\dfrac{d}{dx}\ln(x+2) = \dfrac{1}{u}\dfrac{du}{dx}$

$$\frac{d}{dx}\ln(u) \qquad u = x+2$$

$$f'(x) = \frac{1}{x+2} \qquad \frac{du}{dx} = 1$$

$$x+2 > 0$$

$$x > -2$$

38. $\dfrac{d}{dx}\ln(x+2) = \dfrac{1}{u}\dfrac{du}{dx}$

$$\frac{d}{dx}\ln(u) \qquad u = 2x+2$$

$$\frac{du}{dx} = 2$$

$$= 0 + \frac{d}{dx}\ln x = \frac{1}{x}$$

$$x+1 > 0$$

$$x > -1$$

39. $\dfrac{d}{dx}\ln(2-\cos x) = \dfrac{1}{u}\dfrac{du}{dx}$

$$\frac{d}{dx}\ln(u) \qquad u = 2 - \cos x$$

$$\frac{du}{dx} = \sin x$$

$$f'(x) = \frac{\sin x}{2 - \cos x}$$

Domain of x is all real numbers.

40. $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{1}{u}\dfrac{du}{dx}$

$$\frac{d}{dx}\ln(u) \qquad u = x^2 + 1$$

$$\frac{du}{dx} = 2x$$

$$f'(x) = \frac{2x}{x^2+1}$$

$x^2 + 1 > 0$, Domain of x is all real numbers.

41. $\dfrac{d}{dx}\log_2(3x+1) = \dfrac{1}{4\ln a}\dfrac{du}{dx}$

$$\frac{d}{dx}\log_2(u) \qquad u = 3x+1$$

$$a = 2 \qquad \frac{du}{dx} = 3$$

$$f'(x) = \frac{3}{(3x+1)\ln 2}$$

$$3x+1 > 0$$

$$x > -1/3$$

42. $\dfrac{d}{dx}\log_{10}(\sqrt{x}+1) = \dfrac{1}{u\ln a}\dfrac{du}{dx}$

$$\frac{d}{dx}\log_{10}(u) \qquad u = \sqrt{x}+1$$

$$a = 10 \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}+1}$$

$$f'(x) = \frac{1}{2(\sqrt{x}+1)^2\ln 10}$$

$$= \frac{1}{2(x+1)\ln 10}$$

$$2x+2 > 0$$

$$x > -1$$

43. $y = (\sin x)^x$

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x\ln(\sin x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[x\ln(\sin x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1)$$

$$\frac{dy}{dx} = y\left[x\cot x + \ln(\sin x)\right]$$

$$\frac{dy}{dx} = (\sin x)^x[x\cot x + \ln(\sin x)]$$

44. $y = x^{\tan x}$

$$\ln y = \ln(x^{\tan x})$$

$$\ln y = (\tan x)(\ln x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[(\tan x)(\ln x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x)$$

$$\frac{dy}{dx} = y\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$$

$$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$$

45. $y = \sqrt[5]{\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$

$\ln y = \ln\left(\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$

$\ln y = \dfrac{1}{5}\ln\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}$

$\ln y = \dfrac{1}{5}[4\ln(x-3)+\ln(x^2+1)-3\ln(2x+5)]$

$\dfrac{d}{dx}(\ln y) = \dfrac{4}{5}\dfrac{d}{dx}\ln(x-3)$

$\qquad +\dfrac{1}{5}\dfrac{d}{dx}\ln(x^2+1)-\dfrac{3}{5}\dfrac{d}{dx}\ln(2x+5)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{4}{5}\dfrac{1}{x-3}+\dfrac{1}{5}\dfrac{1}{x^2+1}(2x)-\dfrac{3}{5}\dfrac{1}{2x+5}(2)$

$\dfrac{dy}{dx} = y\left(\dfrac{4}{5(x-3)}+\dfrac{2x}{5(x^2+1)}-\dfrac{6}{5(2x+5)}\right)$

$\dfrac{dy}{dx} = \left(\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \cdot$

$\left(\dfrac{4}{5(x-3)}+\dfrac{2x}{5(x^2+1)}-\dfrac{6}{5(2x+5)}\right)$

46. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}} = \dfrac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$

$\ln y = \ln\dfrac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$

$\ln y = \ln x + \dfrac{1}{2}\ln(x^2+1)-\dfrac{2}{3}\ln(x+1)$

$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}\ln x + \dfrac{1}{2}\dfrac{d}{dx}\ln(x^2+1)-\dfrac{2}{3}\dfrac{d}{dx}\ln(x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}+\dfrac{1}{2}\dfrac{1}{x^2+1}(2x)-\dfrac{2}{3}\dfrac{1}{x+1}(1)$

$\dfrac{dy}{dx} = y\left(\dfrac{1}{x}+\dfrac{x}{x^2+1}-\dfrac{2}{3(x+1)}\right)$

$\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\dfrac{1}{x}+\dfrac{x}{x^2+1}-\dfrac{2}{3(x+1)}\right)$

47. $\qquad y = x^{\ln x}$

$\ln y = \ln x^{\ln x}$

$\ln y = \ln x \, \ln x$

$\dfrac{dy}{dx}(\ln y) = \dfrac{d}{dx}(\ln x \, \ln x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2\ln x}{x}$

$\dfrac{dy}{dx} = \dfrac{2y\ln x}{x} = \dfrac{2x^{\ln x}\ln x}{x}$

48. $\quad y = x^{(1/\ln x)}$

$\ln y = \ln x^{(1/\ln x)}$

$\ln y = \dfrac{\ln x}{\ln x} = 1$

$\dfrac{dy}{dx}(\ln y) = \dfrac{d}{dx}(1)$

$\dfrac{1}{y}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = 0, \ x > 0$

49. The line passes through (a, e^a) for some value of a and has slope $m = e^a$. Since the line also passes through the origin, the slope is also given by e, $e\ln x < x$ or $\ln x^e < x$. and we have $e^a = \dfrac{e^a}{a}$, so $a = 1$. Hence, the slope is e and the equation is $y = ex$.

50. For $y = xe^x$, we have $y' = (x)(e^x)+(e^x)(1) = (x+1)e^x$, so the normal line through the point (a, ae^a) has slope

$m = -\dfrac{1}{(a+1)e^a}$ and its equation is

$y = -\dfrac{1}{(a+1)e^a}(x-a)+ae^a$. The desired normal line includes the point $(0, 0)$, so we have:

$0 = -\dfrac{1}{(a+1)e^a}(0-a)+ae^a$

$0 = \dfrac{a}{(a+1)e^a}+ae^a$

$0 = a\left(\dfrac{1}{(a+1)e^a}+e^a\right)$

$a = 0$ or $\dfrac{1}{(a+1)e^a}+e^a = 0$

The equation $\dfrac{1}{(a+1)e^a}+e^a = 0$ has no solution, so we need to use $a = 0$. The equation of the normal line is

$y = -\dfrac{1}{(0+1)e^0}(x-0)+0e^0$, or $y = -x$.

51. (a) $P(0) = \dfrac{300}{1+2^{4-0}} \approx 18$

(b) for all positive $x \neq e$.

$\qquad = 300\dfrac{d}{dt}(1+2^{4-t})^{-1}$

$\qquad = 300\left(\dfrac{16\ln(2)2^t}{(2^t+16)^2}\right)$

$P'(4) = \dfrac{4800\ln(2)2^4}{(2^4+16)^2} = 52$

(c) After 4 days, the rumor will spread to 52 $\dfrac{\text{students}}{\text{day}}$.

52. (a) $P(0) = \dfrac{200}{1 + e^{5-0}} = 1$

(b) $\dfrac{d}{dt} 200((1 + e^{5-t})^{-1})$

$= 200(-1)(1 + e^{5-t})^{-2} \dfrac{d}{dt}(1 + e^{5-t})$

$= 200(-1)(1 + e^{5-t})^{-2}(e^{5-t})(-1)$

$= \dfrac{200 e^{5-t}}{(1 + e^{5-t})^2}$

$P'(4) = \dfrac{200 e^{5-4}}{(1 + e^{5-4})^2} = 39$

(c) $P'(5) = \dfrac{200 e^{5-5}}{(1 + e^{5-5})^2} = 50$

53. $\dfrac{dA}{dt} = 20 \dfrac{d}{dt}\left(\dfrac{1}{2}\right)^{t/140}$

$= 20 \dfrac{d}{dt} 2^{-t/140}$

$= 20(2^{-t/140})(\ln 2)\dfrac{d}{dt}\left(-\dfrac{t}{140}\right)$

$= 20(2^{-t/140})(\ln 2)\left(-\dfrac{t}{140}\right)$

$= -\dfrac{(2^{-t/140})(\ln 2)}{7}$

At $t = 2$ days, we have $\dfrac{dA}{dt} = -\dfrac{(2^{-1/70})(\ln 2)}{7} \approx -0.098$

grams/day.

This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

54. (a) $\dfrac{d}{dx} \ln(kx) = \dfrac{1}{kx}\dfrac{d}{dx}kx = \dfrac{k}{kx} = \dfrac{1}{x}$

(b) $\dfrac{d}{dx} \ln(kx) = \dfrac{d}{dx}(\ln k + \ln x)$

$= 0 + \dfrac{d}{dx}\ln x = \dfrac{1}{x}$

55. (a) Since $f'(x) = 2^x \ln 2, f'(0) = 2^0 \ln 2 = \ln 2$.

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h \to 0}\dfrac{2^h - 2^0}{h}$

$= \lim\limits_{h \to 0}\dfrac{2^h - 1}{h}$

(c) Since quantities in parts (a) and (b) are equal,

$\lim\limits_{h \to 0}\dfrac{2^h - 1}{h} = \ln 2.$

(d) By following the same procedure as above using

$g(x) = 7^x$, we may see that $\lim\limits_{h \to 0}\dfrac{7^h - 1}{h} = \ln 7.$

56. Recall that a point (a, b) is on the graph of $y = e^x$ if and only if the point (b, a) is on the graph of $y = \ln x$. Since there are points (x, e^x) on the graph of $y = e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large y-coordinates.

57. False. It is $\ln (2)2^x$.

58. False. It is $2e^{2x}$.

59. B. $P(0) = \dfrac{150}{1 + e^{4-t}} = 3$

60. D. $x + 3 > 0$

$x > -3$

61. A. $y = \log_{10}(2x - 3)$

$\dfrac{d}{dx}\log_a u = \dfrac{1}{u \ln a}\dfrac{du}{dx}$

$a = 10$

$u = 2x - 3$

$\dfrac{du}{dx} = 2$

$y' = \dfrac{2}{(2x - 3)\ln 10}$

62. E. $y = 2^{1-x}$

$\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$

$a = 2$

$u = 1 - x$

$\dfrac{du}{dx} = -1$

$y' = 2^{1-x}\ln (2)(-1)$

$y'(2) = -2^{1-2}\ln (2)$

$y'(2) = -\dfrac{\ln (2)}{2}$

63. (a) The graph y_4 is a horizontal line at $y = a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y = \ln a$.

(c) $\dfrac{d}{dx}a^x = a^x$ if and only if $y_3 = \dfrac{y_2}{y_1} = 1$.

So if $y_3 = \ln a$, then $\dfrac{d}{dx}a^x$ with equal a^x if and only if

$\ln a = 1$, or $a = e$.

(d) $y_2 = \dfrac{d}{dx}a^x = a^x \ln a$. This will equal $y_1 = a^x$

if and only if $\ln a = 1$, or $a = e$.

64. $\dfrac{d}{dx}\left(-\dfrac{1}{2}x^2 + k\right) = -x$ and $\dfrac{d}{dx}(\ln x + c) = \dfrac{1}{x}$.

Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

65. (a) Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line

$y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$

for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have

$e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Quick Quiz Sections 3.7–3.9

1. E. $y = \frac{9}{2x} - \frac{x^2}{2}$

$dy = -\frac{9}{2x^2} - x$

$dy = -\frac{9}{2(1)^2} - 1 = -\frac{11}{2}$

2. A. $dy = \frac{d}{dx}(\cos^3(3x-2))$

$dy = -9\cos^2(3x-2)\sin(3x-2)$

3. C. $dy = \frac{d}{dx}(\sin^{-1}(2x))$

$dy = \frac{2}{\sqrt{1-4x^2}}$

4. (a) Differentiate implicitly:

$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$

$1 \cdot y^2 + x \cdot 2y\frac{dy}{dx} - \left(3x^2y + x^3\frac{dy}{dx}\right) = 0$

$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$

$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

(b) If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

at $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)^2(-2) - (-2)^2}{2(1)(-2) - (1)^3} = 2$.

The tangent line is $y + 2 = 2(x - 1)$.

At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)^2(3) - 3^2}{2(1)(3) - 1^3} = 0$.

The tangent line is $y = 3$.

(c) The tangent line is vertical where $2xy - x^3 = 0$, which

implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve

where $x = 0$. If $y = \frac{x^2}{2}$, then $x\left(\frac{x^2}{2}\right) - x^3\left(\frac{x^2}{2}\right) = 6$.

Then the only solution to this equation is $x = \sqrt[5]{-24}$.

Chapter 3 Review Exercises
(pp. 181–184)

1. $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2\sin x \cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2\sin^2 x + 2\cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2\sin x \cos x) = \frac{d}{dx}\sin 2x = (\cos 2x)(2)$

$= 2\cos 2x$

4. $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+1}{2x-1} = \frac{(2x-1)(2) - (2x-1)(2)}{(2x-1)^2} = -\frac{4}{(2x-1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt}\cos(1-2t) = -\sin(1-2t)(-2) = 2\sin(1-2t)$

6. $\frac{ds}{dt} = \frac{d}{dt}\cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2}\csc^2\left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x+1}) = (x)\left(\frac{1}{2\sqrt{2x+1}}\right)(2) + (\sqrt{2x+1})(1)$

$= \frac{x + (2x+1)}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta}\sec(1+3\theta) = \sec(1+3\theta)\tan(1+3\theta)(3)$

$= 3\sec(1+3\theta)\tan(1+3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta}\tan^2(3-\theta^2)$

$= 2\tan(3-\theta^2)\frac{d}{d\theta}\tan(3-\theta^2)$

$= 2\tan(3-\theta^2)\sec^2(3-\theta^2)(-2\theta)$

$= -4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2\csc 5x)$

$= (x^2)(-\csc 5x \cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2\csc 5x \cot 5x + 2x\csc 5x$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln\sqrt{x} = \dfrac{1}{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2x}, \; x > 0$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(1+e^x) = \dfrac{1}{1+e^x}\dfrac{d}{dx}(1+e^x) = \dfrac{e^x}{1+e^x}$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}(xe^{-x}) = (x)(e^{-x})(-1)+(e^{-x})(1) = -xe^{-x}+e^{-x}$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{1+\ln x}) = \dfrac{d}{dx}(e^1 e^{\ln x}) = \dfrac{d}{dx}(ex) = e$

16. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(\sin x) = \dfrac{1}{\sin x}\dfrac{d}{dx}(\sin x) = \dfrac{\cos x}{\sin x} = \cot x$, for values
of x in the intervals $(k\pi, \; (k+1)\pi)$, where k is even.

17. $\dfrac{dr}{dx} = \dfrac{d}{dx}\ln(\cos^{-1} x) = \dfrac{1}{\cos^{-1} x}\dfrac{d}{dx}\cos^{-1} x$

$= \dfrac{1}{\cos^{-1} x}\left(-\dfrac{1}{\sqrt{1-x^2}}\right) = -\dfrac{1}{\cos^{-1} x\sqrt{1-x^2}}$

18. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\log_2(\theta^2) = \dfrac{1}{\theta^2\ln 2}\dfrac{d}{d\theta}(\theta^2) = \dfrac{2\theta}{\theta^2\ln 2} = \dfrac{2}{\theta\ln 2}$

19. $\dfrac{ds}{dt} = \dfrac{d}{dt}\log_5(t-7) = \dfrac{1}{(t-7)\ln 5}\dfrac{d}{dt}(t-7) = \dfrac{1}{(t-7)\ln 5}$,
$t > 7$

20. $\dfrac{ds}{dt} = \dfrac{d}{dt}(8^{-t}) = 8^{-t}(\ln 8)\dfrac{d}{dt}(-t) = -8^{-t}\ln 8$

21. Use logarithmic differentiation.

$y = x^{\ln x}$
$\ln y = \ln(x^{\ln x})$
$\ln y = (\ln x)(\ln x)$
$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}(\ln x)^2$
$\dfrac{1}{y}\dfrac{dy}{dx} = 2\ln x\dfrac{d}{dx}\ln x$
$\dfrac{dy}{dx} = \dfrac{2y\ln x}{x}$
$\dfrac{dy}{dx} = \dfrac{2x^{\ln x}\ln x}{x}$

22. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{(2x)2^x}{\sqrt{x^2+1}}$

$= \dfrac{\sqrt{x^2+1}\dfrac{d}{dx}\left[(2x)2^x\right] - (2x)(2^x)\dfrac{d}{dx}\sqrt{x^2+1}}{x^2+1}$

$= \dfrac{\sqrt{x^2+1}\left[(2x)(2^x)(\ln 2)+(2^x)(2)\right] - (2x)(2^x)\dfrac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$

$= \dfrac{(x^2+1)(2^x)(2x\ln 2+2)-2x^2(2^x)}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)[(x^2+1)(x\ln 2+1)-x^2]}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)(x^3\ln 2+x^2+x\ln 2+1-x^2)}{(x^2+1)^{3/2}}$

$= \dfrac{(2\cdot 2^x)(x^3\ln 2+x\ln 2+1)}{(x^2+1)^{3/2}}$

Alternate solution, using logarithmic differentiation:

$y = \dfrac{(2x)2^x}{\sqrt{x^2+1}}$

$\ln y = (2x)+\ln(2^x)-\ln\sqrt{x^2+1}$

$\ln y = \ln 2+\ln x+x\ln 2-\dfrac{1}{2}\ln(x^2+1)$

$\dfrac{d}{dx}\ln y = \dfrac{d}{dx}[\ln 2+\ln x+x\ln 2-\dfrac{1}{2}\ln(x^2+1)]$

$\dfrac{1}{y}\dfrac{dy}{dx} = 0+\dfrac{1}{x}+\ln 2-\dfrac{1}{2}\dfrac{1}{x^2+1}(2x)$

$\dfrac{dy}{dx} = y\left(\dfrac{1}{x}+\ln 2-\dfrac{x}{x^2+1}\right)$

$\dfrac{dy}{dx} = \dfrac{(2x)2^x}{\sqrt{x^2+1}}\left(\dfrac{1}{x}+\ln 2-\dfrac{x}{x^2+1}\right)$

23. $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{\tan^{-1} x} = e^{\tan^{-1} x}\dfrac{d}{dx}\tan^{-1} x = \dfrac{e^{\tan^{-1} x}}{1+x^2}$

24. $\dfrac{dy}{du} = \dfrac{d}{dx}\sin^{-1}\sqrt{1-u^2}$

$= \dfrac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\dfrac{d}{du}\sqrt{1-u^2}$

$= \dfrac{1}{\sqrt{u^2}}\dfrac{1}{2\sqrt{1-u^2}}(-2u) = \dfrac{u}{|u|\sqrt{1-u^2}}$

25. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(t\sec^{-1} t-\dfrac{1}{2}\ln t\right)$

$= (t)\left(\dfrac{1}{|t|\sqrt{t^2-1}}\right)+(\sec^{-1} t)(1)-\dfrac{1}{2t}$

$= \dfrac{t}{|t|\sqrt{t^2-1}}+\sec^{-1} t-\dfrac{1}{2t}$

26. $\dfrac{dy}{dt} = \dfrac{d}{dt}[(1+t^2)\cot^{-1} 2t]$

$= (1+t^2)\left(-\dfrac{1}{1+(2t)^2}\right)(2)+(\cot^{-1} 2t)(2t)$

$= -\dfrac{2+2t^2}{1+4t^2}+2t\cot^{-1} 2t$

27. $\dfrac{dy}{dz} = \dfrac{d}{dz}(z\cos^{-1} z - \sqrt{1-z^2})$

$= (z)\left(-\dfrac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1} z)(1) - \dfrac{1}{2\sqrt{1-z^2}}(-2z)$

$= -\dfrac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \dfrac{z}{\sqrt{1-z^2}}$

$= \cos^{-1} z$

28. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2\sqrt{x-1}\,\csc^{-1}\sqrt{x})$

$= (2\sqrt{x-1})\left(-\dfrac{1}{\left|\sqrt{x}\right|(\sqrt{x})^2 - 1}\right)\left(\dfrac{1}{2\sqrt{x}}\right)$

$\quad + (2\csc^{-1}\sqrt{x})\left(\dfrac{1}{2\sqrt{x-1}}\right)$

$= -\dfrac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \dfrac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

$= -\dfrac{1}{x} + \dfrac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

29. $\dfrac{dy}{dx} = \dfrac{d}{dx}\csc^{-1}(\sec x)$

$= \left(-\dfrac{1}{\left|\sec x\right|\sqrt{\sec^2 x - 1}}\right)\dfrac{d}{dx}(\sec x)$

$= -\dfrac{1}{\left|\sec x\right|\sqrt{\tan^2 x - 1}}\sec x\tan x$

$= -\dfrac{\sec x\tan x}{\left|\sec x\tan x\right|}$

$= -\dfrac{\dfrac{1}{\cos x}\dfrac{\sin x}{\cos x}}{\left|\dfrac{1}{\cos x}\dfrac{\sin x}{\cos x}\right|} = -\dfrac{\sin x}{\left|\sin x\right|}$

$= \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \dfrac{\pi}{2} \\[2mm] 1, & \pi < x \le 2\pi, \quad x \ne \dfrac{3\pi}{2} \end{cases}$

Alternate method:

On the domain $0 \le x \le 2\pi, x \ne \dfrac{\pi}{2}, x \ne \dfrac{3\pi}{2}$, we may rewrite the function as follows:

$y = \csc^{-1}(\sec x)$

$= \dfrac{\pi}{2} - \sec^{-1}(\sec x)$

$= \dfrac{\pi}{2} - \cos^{-1}(\cos x)$

$= \begin{cases} \dfrac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \dfrac{\pi}{2} \\[2mm] \dfrac{\pi}{2} - (\pi - x), & \pi < x \le 2\pi, \quad x \ne \dfrac{3\pi}{2} \end{cases}$

$= \begin{cases} \dfrac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \dfrac{\pi}{2} \\[2mm] \dfrac{\pi}{2} - (\pi + x), & \pi < x \le 2\pi, \quad x \ne \dfrac{3\pi}{2} \end{cases}$

Therefore, $\dfrac{dy}{dx} = \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \dfrac{\pi}{2} \\[2mm] 1, & \pi < x \le 2\pi, \quad x \ne \dfrac{3\pi}{2} \end{cases}$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

30. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)^2$

$= 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{(1-\cos\theta)(\cos\theta) - (1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}\right)$

$= 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1-\cos\theta)^2}\right)$

$= 2\left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)\left(\dfrac{\cos\theta - \sin\theta - 1}{(1-\cos\theta)^2}\right)$

31. Since $y = \ln x^2$ is defined for all

$x \ne 0$ and $\dfrac{dy}{dx} = \dfrac{1}{x^2}\dfrac{d}{dx}(x^2) = \dfrac{2x}{x^2} = \dfrac{2}{x}$, the function is differentiable for all $x \ne 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and

$\dfrac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$, the function is differentiable for all real x.

33. Since $y = \sqrt{\dfrac{1-x}{1+x^2}}$ is defined for all $x < 1$ and

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\dfrac{1-x}{1+x^2}}}\cdot\dfrac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$

$= \dfrac{x^2 - 2x - 1}{2\sqrt{1-x}(1+x^2)^{3/2}}$, which is defined only for $x < 1$, the function is differentiable for all $x < 1$.

34. Since $y = (2x-7)^{-1}(x+5) = \dfrac{x+5}{2x-7}$ is defined for all

$x \ne \dfrac{7}{2}$ and $\dfrac{dy}{dx} = \dfrac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2} = -\dfrac{17}{(2x-7)^2}$,

the function is differentiable for all $x \ne \dfrac{7}{2}$.

35. Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$\dfrac{dy}{dx}(xy) + \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(3y) = \dfrac{d}{dx}(1)$

$x\dfrac{d}{dx} + (y)(1) + 2 + 3\dfrac{dy}{dx} = 0$

$(x+3)\dfrac{dy}{dx} = -(y+2)$

$\dfrac{dy}{dx} = -\dfrac{y+2}{x+3}$

36. Use implicit differentiation.

$$5x^{4/5} + 10y^{6/5} = 15$$

$$\frac{d}{dx}(5x^{4/5}) + \frac{d}{dx}(10y^{6/5}) = \frac{d}{dx}(15)$$

$$4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^{-1/5}}{12y^{1/5}} = -\frac{1}{3(xy)^{1/5}}$$

37. Using implicit differentiation.

$$\sqrt{xy} = 1$$

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(1)$$

$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx} + (y)(1)\right] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Alternate method:

38. Use implicit differentiation.

$$y^2 = \frac{x}{x+1}$$

$$\frac{d}{dx}y^2 = \frac{d}{dx}\frac{x}{x+1}$$

$$2y\frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2y(x+1)^2}$$

39.

$$x^3 + y^3 = 1$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(1)$$

$$3x^2 + 3y^2 y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

$$y'' = \frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$

$$= -\frac{(y^2)(2x) - (x^2)(2y)(y')}{y^4}$$

$$= -\frac{(y^2)(2x) - (x^2)(2y)\left(-\dfrac{x^2}{y^2}\right)}{y^4}$$

$$= -\frac{2xy^3 + 2x^4}{y^5}$$

$$= -\frac{2x(x^3 + y^3)}{y^5}$$

$$= -\frac{2x}{y^5}$$

since $x^3 + y^3 = 1$

40.

$$y^2 = 1 - \frac{2}{x}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{2}{x}\right)$$

$$2yy' = \frac{2}{x^2}$$

$$y' = \frac{2}{x^2(2y)} = \frac{1}{x^2 y}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x^2 y}\right)$$

$$= -\frac{1}{(x^2 y)^2}\frac{d}{dx}(x^2 y)$$

$$= -\frac{1}{(x^2 y)^2}[(x^2)(y') + (y)(2x)]$$

$$= -\frac{1}{(x^2 y)^2}\left[(x^2)\left(\frac{1}{x^2 y}\right) + 2xy\right]$$

$$= -\frac{1}{x^4 y^2}\left(\frac{1}{y} + 2xy\right)$$

$$= -\frac{1 + 2xy^2}{x^4 y^3}$$

41.

$$y^3 + y = 2\cos x$$

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(y) = \frac{d}{dx}(2\cos x)$$

$$3y^2 y' + y' = -2\sin x$$

$$(3y^2 + 1)y' = -2\sin x$$

$$y' = -\frac{2\sin x}{3y^2 + 1}$$

$$y'' = \frac{d}{dx}\left(-\frac{2\sin x}{3y^2 + 1}\right)$$

$$= -\frac{(3y^2 + 1)(2\cos x) - (2\sin x)(6yy')}{(3y^2 + 1)^2}$$

$$= -\frac{(3y^2 + 1)(2\cos x) - (12y\sin x)\left(-\dfrac{2\sin x}{3y^2 + 1}\right)}{(3y^2 + 1)^2}$$

$$= -2\frac{(3y^2 + 1)^2 \cos x + 12y\sin^2 x}{(3y^2 + 1)^3}$$

42. $$x^{1/3} + y^{1/3} = 4$$

$$\frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(y^{1/3}) = \frac{d}{dx}(4)$$

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y' = -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$

$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy' - (y)(1)}{x^2}\right)$$

$$= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right] - y}{x^2}\right)$$

$$= -\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3} - x^{-2}y)$$

$$= \frac{2}{3}x^{-4/3}y^{1/3} + \frac{2}{3}x^{-5/3}y^{2/3}$$

43. $y' = 2x^3 - 3x - 1,$
$y'' = 6x^2 - 3,$
$y''' = 12x,$
$y^{(4)} = 12,$ and the rest are all zero.

44. $y' = \dfrac{x^4}{24},$

$y'' = \dfrac{x^3}{6},$

$6y''' = \dfrac{x^2}{2},$

$y^{(4)} = x,$

$y^{(5)} = 1,$ and the rest are all zero.

45. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt{x^2 - 2x} = \dfrac{1}{2\sqrt{x^2-2x}}(2x-2) = \dfrac{x-1}{\sqrt{x^2-2x}}$

At $x = 3$, we have $y = \sqrt{3^2 - 2(3)} = \sqrt{3}$

and $\dfrac{dy}{dx} = \dfrac{3-1}{\sqrt{3^2-2(3)}} = \dfrac{2}{\sqrt{3}}.$

(a) Tangent: $y = \dfrac{2}{\sqrt{3}}(x-3) + \sqrt{3}$ or $y = \dfrac{2}{\sqrt{3}}x - \sqrt{3}$

(b) Normal: $y = -\dfrac{\sqrt{3}}{2}(x-3) + \sqrt{3}$ or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{5\sqrt{3}}{2}$

46. $\dfrac{dy}{dx} = \dfrac{d}{dx}(4 + \cot x - 2\csc x)$

$\quad = -\csc^2 x + 2\csc x \cot x$

At $x = \dfrac{\pi}{2}$, we have

$y = 4 + \cot\dfrac{\pi}{2} - 2\csc\dfrac{\pi}{2} = 4 + 0 - 2 = 2$ and

$\dfrac{dy}{dx} = -\csc^2\dfrac{\pi}{2} + 2\csc\dfrac{\pi}{2}\cot\dfrac{\pi}{2} = -1 + 2(1)(0) = -1.$

(a) Tangent: $y = -1\left(x - \dfrac{\pi}{2}\right) + 2$ or $y = -x + \dfrac{\pi}{2} + 2$

(b) Normal: $y = -1\left(x - \dfrac{\pi}{2}\right) + 2$ or $y = x - \dfrac{\pi}{2} + 2$

47. Use implicit differentiation.

$$x^2 + 2y^2 = 9$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(9)$$

$$2x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$$

Slope at $(1, 2)$: $-\dfrac{1}{2(2)} = -\dfrac{1}{4}$

(a) Tangent: $y = -\dfrac{1}{4}(x-1) + 2$ or $y = -\dfrac{1}{4}x + \dfrac{9}{4}$

(b) Normal: $y = 4(x-1) + 2$ or $y = 4x - 2$

48. Use implicit differentiation.

$$x + \sqrt{xy} = 6$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}(6)$$

$$1 + \frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right) + (y)(1)\right] = 0$$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx} = -1 - \frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{xy}}{x}\left(-1 - \frac{y}{2\sqrt{xy}}\right)$$

$$= -2\sqrt{\frac{y}{x}} - \frac{y}{x}$$

Slope at $(4, 1)$: $-2\sqrt{\dfrac{1}{4}} - \dfrac{1}{4} = -\dfrac{2}{2} - \dfrac{1}{4} = -\dfrac{5}{4}$

(a) Tangent: $y = -\dfrac{5}{4}(x-4) + 1$ or $y = -\dfrac{5}{4}x + 6$

(b) Normal: $y = -\dfrac{4}{5}(x-4) + 1$ or $y = -\dfrac{4}{5}x - \dfrac{11}{5}$

49. $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{-2\sin t}{2\cos t}=-\tan t$

At $t=\dfrac{3\pi}{4}$, we have $x=2\sin\dfrac{3\pi}{4}=\sqrt{2},$

$y=2\cos\dfrac{3\pi}{4}=-\sqrt{2},$ and $\dfrac{dy}{dx}=-\tan\dfrac{3\pi}{4}=1.$

The equation of the tangent line is

$y=1(x-\sqrt{2})+(-\sqrt{2}),$ or $y=x-2\sqrt{2}.$

50. $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{4\cos t}{-3\sin t}=-\dfrac{4}{3}\cot t$

At $t=\dfrac{3\pi}{4},$ we have $x=3\cos\dfrac{3\pi}{4}=-\dfrac{3\sqrt{2}}{2},$

$y=4\sin\dfrac{3\pi}{4}=2\sqrt{2},$ and $\dfrac{dy}{dx}=-\dfrac{4}{3}\cot\dfrac{3\pi}{4}=\dfrac{4}{3}.$

The equation of the tangent line is

$y=\dfrac{4}{3}\left(x+\dfrac{3\sqrt{2}}{2}\right)+2\sqrt{2},$ or $y=\dfrac{4}{3}x+4\sqrt{2}.$

51. $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{5\sec^2 t}{3\sec t\tan t}=\dfrac{5\sec t}{3\tan t}=\dfrac{5}{3\sin t}$

At $t=\dfrac{\pi}{6},$ we have $x=3\sec\dfrac{\pi}{6}=2\sqrt{3},$

$y=5\tan\dfrac{\pi}{6}=\dfrac{5\sqrt{3}}{3},$ and $\dfrac{dy}{dx}=\dfrac{5}{3\sin\left(\dfrac{\pi}{6}\right)}=\dfrac{10}{3}.$

The equation of the tangent line is

$y=\dfrac{10}{3}(x-2\sqrt{3})+\dfrac{5\sqrt{3}}{3},$ or $y=\dfrac{10}{3}x-5\sqrt{3}.$

52. $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{1+\cos t}{-\sin t}$

At $t=-\dfrac{\pi}{4},$ we have $x=\cos\left(-\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2},$

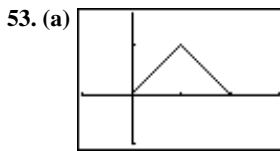$y=-\dfrac{\pi}{4}+\sin\left(-\dfrac{\pi}{4}\right)=-\dfrac{\pi}{4}-\dfrac{\sqrt{2}}{2},$ and

$\dfrac{dy}{dx}=\dfrac{1+\cos\left(-\dfrac{\pi}{4}\right)}{-\sin\left(-\dfrac{\pi}{4}\right)}=\dfrac{1+\dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{2}}{2}}=\sqrt{2}+1.$

The equation of the tangent line is

$y=(\sqrt{2}+1)\left(x-\dfrac{\sqrt{2}}{2}\right)-\dfrac{\pi}{4}-\dfrac{\sqrt{2}}{2},$ or

$y=(1+\sqrt{2})x-\sqrt{2}-1-\dfrac{\pi}{4}.$

This is approximately $y=2.414x-3.200.$

53. (a)

[-1, 3] by [-1, 5/3]

(b) Yes, because both of the one-sided limits as $x\to 1$ are equal to $f(1)=1.$

(c) No, because the left-hand derivative at $x=1$ is $+1$ and the right-hand derivative at $x=1$ is $-1.$

54. (a) The function is continuous for all values of $m,$ because the right-hand limit as $x\to 0$ is equal to $f(0)=0$ for any value of $m.$

(b) The left-hand derivative at $x=0$ is $2\cos(2\bullet 0)=2,$ and the right-hand derivative at $x=0$ is $m,$ so in order for the function to be differentiable at $x=0,$ m must be 2.

55. (a) For all $x\neq 0$ **(b)** At $x=0$

(c) Nowhere

56. (a) For all x **(b)** Nowhere

(c) Nowhere

57. Note that $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}(2x-3)=-3$ and

$\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}(x-3)=-3.$ Since these values agree with $f(0),$ the function is continuous at $x=0.$ On the other hand,

$f'(x)=\begin{cases}2, & -1\le x<0\\ 1, & 0<x\le 4\end{cases},$ so the derivative is undefined at $x=0.$

(a) $[-1,0)\cup(0,4]$ **(b)** At $x=0$

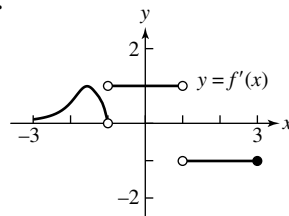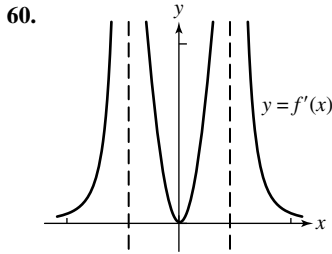**(c)** Nowhere in its domain

58. Note that the function is undefined at $x=0.$

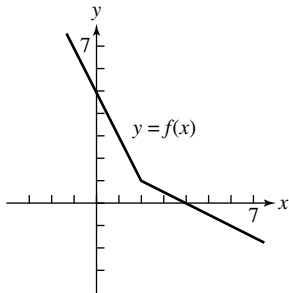**(a)** $[-2,0)\cup(0,2]$ **(b)** Nowhere

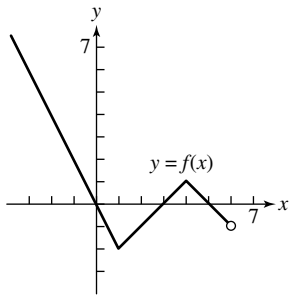**(c)** Nowhere in its domain

59.

60.

$y = f'(x)$

61. (a) iii   **(b)** i

(c) ii

62. The graph passes through (0, 5) and has slope –2 for $x < 2$ and slope –0.5 for $x > 2$.

$y = f(x)$

63. The graph passes through (–1, 2) and has slope –2 for $x < 1$, slope 1 for $1 < x < 4$, and slope –1 for $4 < x < 6$.


$y = f(x)$

64. i. If $f(x) = \dfrac{9}{28}x^{7/3} + 9$, then $f'(x) = \dfrac{3}{4}x^{4/3}$ and

$f''(x) = x^{1/3}$, which matches the given equation.

ii. If $f'(x) = \dfrac{9}{28}x^{7/3} - 2$, then $f''(x) = \dfrac{3}{4}x^{4/3}$, which

contradicts the given equation $f''(x) = x^{1/3}$.

iii. If $f'(x) = \dfrac{3}{4}x^{4/3} + 6$, then $f''(x) = x^{1/3}$, which matches

the given equation.

iv. If $f(x) = \dfrac{3}{4}x^{4/3} - 4$, then $f'(x) = x^{1/3}$ and

$f''(x) = \dfrac{1}{3}x^{-2/3}$, which contradicts the given equation

$f''(x) = x^{1/3}$.

Answer is D: i and iii only could be true. Note, however that i and iii could not simultaneously be true.

65. (a)

[-1, 5] by [-10,80]
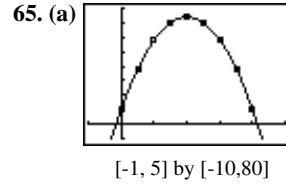
(b) *t* interval

t interval	avg. vel.
[0, 0.5]	$\dfrac{38-10}{0.5-0} = 56$
[0.5, 1]	$\dfrac{58-38}{1-0.5} = 40$
[1, 1.5]	$\dfrac{70-58}{1.5-1} = 24$
[1.5, 2]	$\dfrac{74-70}{2-1.5} = 8$
[2, 2.5]	$\dfrac{70-74}{2.5-2} = -8$
[2.5, 3]	$\dfrac{58-70}{3-2.5} = -24$
[3, 3.5]	$\dfrac{38-58}{3.5-3} = -40$
[3.5, 4]	$\dfrac{10-38}{4-3.5} = -56$

(c)

[-1, 5] by [-80, 80]
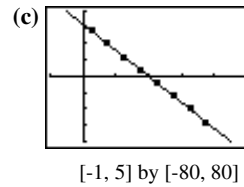
(d) Average velocity is a good approximation to velocity.

66. (a) $\dfrac{d}{dx}\left[\sqrt{x}\,f(x)\right] = \sqrt{x}\,f'(x) + \dfrac{1}{2\sqrt{x}}f(x)$

At $x = 1$, the derivative is

$\sqrt{1}f'(1) + \dfrac{1}{2\sqrt{1}}f(1) = 1\left(\dfrac{1}{5}\right) + \left(\dfrac{1}{2}\right)(-3) = -\dfrac{13}{10}.$

(b) $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2\sqrt{f(x)}}f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is $\dfrac{f'(0)}{2\sqrt{f(0)}} = -\dfrac{2}{2\sqrt{9}} = -\dfrac{1}{3}.$

(c) $\dfrac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\dfrac{d}{dx}\sqrt{x} = \dfrac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is $\dfrac{f'(\sqrt{1})}{2\sqrt{1}} = \dfrac{f'(1)}{2} = \dfrac{\frac{1}{5}}{2} = \dfrac{1}{10}.$

(d) $\dfrac{d}{dx}f(1 - 5\tan x) = f'(1 - 5\tan x)(-5\sec^2 x)$

At $x = 0$, the derivative is

$f'(1 - 5\tan 0)(-5\sec^2 0) = f'(1)(-5) = \left(\dfrac{1}{5}\right)(-5) = -1.$

66. Continued

(e) $\dfrac{d}{dx}\dfrac{f(x)}{2+\cos x}=\dfrac{(2+\cos x)(f'(x))-(f(x))(-\sin x)}{(2+\cos x)^2}$

At $x=0$, the derivative is

$\dfrac{(2+\cos 0)(f'(0))-(f(0))(-\sin 0)}{(2+\cos 0)^2}=\dfrac{3f'(0)}{3^2}=-\dfrac{2}{3}.$

(f) $\dfrac{d}{dx}[10\sin\left(\dfrac{\pi x}{2}\right)f^2(x)]$

$=10\left(\sin\dfrac{\pi x}{2}\right)(2f(x)f'(x))+10f^2(x)\left(\cos\dfrac{\pi x}{2}\right)\left(\dfrac{\pi}{2}\right)$

$=20f(x)f'(x)\sin\dfrac{\pi x}{2}+5\pi f^2(x)\cos\dfrac{\pi x}{2}$

At $x=1$, the derivative is

$20f(1)f'(1)\sin\dfrac{\pi}{2}+5\pi f^2(1)\cos\dfrac{\pi}{2}$

$=20(-3)\left(\dfrac{1}{5}\right)(1)+5\pi(-3)^2(0)$

$=-12.$

67. (a) $\dfrac{d}{dx}\big[3f(x)-g(x)\big]=3f'(x)-g'(x)$

At $x=-1$, the derivative is
$3f'(-1)-g'(-1)=3(2)-1=5.$

(b) $\dfrac{d}{dx}\Big[f^2(x)g^3(x)\Big]$

$=f^2(x)\cdot 3g^2(x)g'(x)+g^3(x)\cdot 2f(x)f'(x)$
$=f(x)g^2(x)\big[3f(x)g'(x)+2g(x)f'(x)\big]$

At $x=0$, the derivative is

$f(0)g^2(0)\big[3f(0)g'(0)+2g(0)f'(0)\big]$
$=(-1)(-3)^2\big[3(-1)(4)+2(-3)(-2)\big]$
$=-9\big[-12+12\big]=0.$

(c) $\dfrac{d}{dx}g(f(x))=g'(f(x))f'(x)$

At $x=-1$, the derivative is
$g'(f(-1))f'(-1)=g'(0)f'(-1)=(4)(2)=8.$

(d) $\dfrac{d}{dx}f(g(x))=f'(g(x))g'(x)$

At $x=-1$, the derivative is
$f'(g(-1))g'(-1)=f'(-1)g'(-1)=(2)(1)=2.$

(e) $\dfrac{d}{dx}\dfrac{f(x)}{g(x)+2}=\dfrac{(g(x)+2)f'(x)-f(x)g'(x)}{(g(x)+2)^2}$

At $x=0$, the derivative is

$\dfrac{(g(0)+2)f'(0)-f(0)g'(0)}{(g(0)+2)^2}=\dfrac{(-3+2)(-2)-(-1)(4)}{(-3+2)^2}$

$=6.$

(f) $\dfrac{d}{dx}g(x+f(x))=g'(x+f(x))\dfrac{d}{dx}(x+f(x))$

$=g'(x+f(x))(1+f'(x))$

At $x=0$, the derivative is $g'(0+f(0))[1+f'(0)]$

$=g'(0-1)[1+(-2)]=(1)(-1)=-1$

68. $\dfrac{dw}{ds}=\dfrac{dw}{dr}\dfrac{dr}{ds}=\dfrac{d}{dr}[\sin(\sqrt{r}-2)]\dfrac{d}{ds}\left[8\sin\left(s+\dfrac{\pi}{6}\right)\right]$

$=\left[\cos(\sqrt{r}-2)\dfrac{1}{2\sqrt{r}}\right]\left[8\cos\left(s+\dfrac{\pi}{6}\right)\right]$

At $s=0$, we have $r=8\sin\left(0+\dfrac{\pi}{6}\right)=4$ and so

$\dfrac{dw}{ds}=\left[\cos(\sqrt{4}-2)\dfrac{1}{2\sqrt{4}}\right]\left[8\cos\left(0+\dfrac{\pi}{6}\right)\right]$

$=\left(\dfrac{\cos 0}{4}\right)\left(8\cos\dfrac{\pi}{6}\right)=\left(\dfrac{1}{4}\right)\left(\dfrac{8\sqrt{3}}{2}\right)=\sqrt{3}$

69. Solving $\theta^2 t+\theta=1$ for t, we have

$t=\dfrac{1-\theta}{\theta^2}=\theta^{-2}-\theta^{-1}$, and we may write:

$$\dfrac{dr}{d\theta}=\dfrac{dr}{dt}\dfrac{dt}{d\theta}$$

$$\dfrac{d}{d\theta}(\theta^2+7)^{1/3}=\dfrac{dr}{dt}\dfrac{d}{d\theta}(\theta^{-2}-\theta^{-1})$$

$$\dfrac{1}{3}(\theta^2+7)^{-2/3}(2\theta)=\left(\dfrac{dr}{dt}\right)(-2\theta^{-3}+\theta^{-2})$$

$$\dfrac{dr}{dt}=\dfrac{2\theta(\theta^2+7)^{-2/3}}{3(-2\theta^{-3}+\theta^{-2})}=\dfrac{2\theta^4(\theta^2+7)^{-2/3}}{3(\theta-2)}$$

At $t=0$, we may solve $\theta^2 t+\theta=1$ to obtain $\theta=1$,

and so $\dfrac{dr}{dt}=\dfrac{2(1)^4(1^2+7)^{-2/3}}{3(1-2)}=\dfrac{2(8)^{-2/3}}{-3}=-\dfrac{1}{6}.$

70. (a) One possible answer:

$$x(t)=10\cos\left(t+\dfrac{\pi}{4}\right),\ y(t)=1$$

(b) $s(0)=10\cos\dfrac{\pi}{4}=5\sqrt{2}$

(c) Farthest left:

When $\cos\left(t+\dfrac{\pi}{4}\right)=-1$, we have $s(t)=-10.$

Farthest right:

When $\cos\left(t+\dfrac{\pi}{4}\right)=1$, we have $s(t)=10.$

70. Continued

(d) Since $\cos\dfrac{\pi}{2} = 0$, the particle first reaches the origin at

$t = \dfrac{\pi}{4}$. The velocity is given by $v(t) = -10\sin\left(t + \dfrac{\pi}{4}\right)$,

so the velocity at $t = \dfrac{\pi}{4}$ is $-10\sin\dfrac{\pi}{2} = -10$, and the speed

at $t = \dfrac{\pi}{4}$ is $|-10| = 10$. The acceleration is given by

$a(t) = -10\cos\left(t + \dfrac{\pi}{4}\right)$, so the acceleration at

$t = \dfrac{\pi}{4}$ is $-10\cos\dfrac{\pi}{2} = 0$.

71. (a) $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 16t^2) = 64 - 32t$

$\dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}(64 - 32t) = -32$

(b) The maximum height is reached when $\dfrac{ds}{dt} = 0$, which

occurs at $t = 2$ sec.

(c) When $t = 0, \dfrac{ds}{dt} = 64$, so the velocity is 64 ft/sec.

(d) Since $\dfrac{ds}{dt} = \dfrac{d}{dt}(64t - 2.6t^2) = 64 - 5.2t$, the maximum

height is reached at $t = \dfrac{64}{5.2} \approx 12.3$ sec. The maximum

height is $s\left(\dfrac{64}{5.2}\right) \approx 393.8$ ft.

72. (a) Solving $160 = 490\,t^2$, it takes $\dfrac{4}{7}$ sec. The average

velocity is $\dfrac{160}{4} = 280$ cm/sec.

(b) Since $v(t) = \dfrac{ds}{dt} = 980t$, the velocity is $(980)\left(\dfrac{4}{7}\right) = 560$

cm/sec. Since $a(t) = \dfrac{dv}{dt} = 980$, the acceleration is

980 cm / sec^2.

73. $\dfrac{dV}{dx} = \dfrac{d}{dx}\left[\pi\left(10 - \dfrac{x}{3}\right)x^2\right] = \dfrac{d}{dx}\left[\pi\left(10x^2 - \dfrac{1}{3}x^3\right)\right]$

$= \pi(20x - x^2)$

74. (a) $r(x) = \left(3 - \dfrac{x}{40}\right)^2 x = 9x - \dfrac{3}{20}x^2 + \dfrac{1}{1600}x^3$

(b) The marginal revenue is

$r'(x) = 9 - \dfrac{3}{10}x + \dfrac{3}{1600}x^2$

$= \dfrac{3}{1600}(x^2 - 160x + 4800)$

$= \dfrac{3}{1600}(x - 40)(x - 120)$,

which is zero when $x = 40$ or $x = 120$. Since the bus
holds only 60 people, we require $0 \le x \le 60$. The
marginal revenue is 0 when there are 40 people, and the

corresponding fare is $p(40) = \left(3 - \dfrac{40}{40}\right)^2 = \$4.00$.

(c) One possible answer:
If the current ridership is less than 40, then the proposed
plan may be good. If the current ridership is greater than
or equal to 40, then the plan is not a good idea. Look at
the graph of $y = r(x)$.
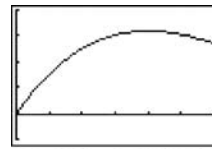
[0, 60] by [−50, 200]

75. (a) Since $x = \tan\theta$, we have

$\dfrac{dx}{dt} = (\sec^2\theta)\dfrac{d\theta}{dt} = -0.6\sec^2\theta$. At point A, we have

$\theta = 0$ and $\dfrac{dx}{dt} = -0.6\sec^2 0 = -0.6$ km/sec.

(b) $0.6\,\dfrac{\text{rad}}{\text{sec}} \cdot \dfrac{1\text{ revolution}}{2\pi\text{ rad}} \cdot \dfrac{60\text{ sec}}{1\text{ min}} = \dfrac{18}{\pi}$ revolutions per

minute or approximately 5.73 revolutions per minute.

76. Let $f(x) = \sin(x - \sin x)$. Then

$f'(x) = \cos(x - \sin x)\dfrac{d}{dx}(x - \sin x)$

$= \cos(x - \sin x)(1 - \cos x)$. This derivative is zero when

$\cos(x - \sin x) = 0$ (which we need not solve) or

when $\cos x = 1$, which occurs at $x = 2k\pi$ for integers k. For

each of these values, $f(x) = f(2k\pi) = \sin(2k\pi - \sin 2k\pi) =$

$\sin(2k\pi - 0) = 0$. Thus, $f(x) = f'(x) = 0$ for $x = 2k\pi$,

which means that the graph has a horizontal tangent at each
of these values of x.

77. $y'(r) = \dfrac{d}{dr}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2l}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dr}\left(\dfrac{1}{r}\right) = -\dfrac{1}{2r^2 l}\sqrt{\dfrac{T}{\pi d}}$

$y'(l) = \dfrac{d}{dr}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2r}\sqrt{\dfrac{T}{\pi d}}\right)\dfrac{d}{dr}\left(\dfrac{1}{l}\right) = -\dfrac{1}{2r^2 l}\sqrt{\dfrac{T}{\pi d}}$

$y'(d) = \dfrac{d}{dd}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\right)\dfrac{d}{dd}(d^{-1/2})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi}}\left(-\dfrac{1}{2}d^{-3/2}\right) = -\dfrac{1}{4rl}\sqrt{\dfrac{T}{\pi d^3}}$

$y'(T) = \dfrac{d}{dT}\left(\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi d}}\right) = \left(\dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\right)\dfrac{d}{dt}(\sqrt{T})$

$= \dfrac{1}{2rl}\sqrt{\dfrac{1}{\pi d}}\left(\dfrac{1}{2\sqrt{T}}\right) = \dfrac{1}{4rk\sqrt{\pi dT}}$

Since $y'(r) < 0$, $y'(l) < 0$, and $y'(d) < 0$, increasing r, l, or d would decrease the frequency. Since $y'(T) > 0$, increasing T would increase the frequency.

78. (a) $P(0) = \dfrac{200}{1+e^5} \approx 1$ student

(b) $\displaystyle\lim_{h\to\infty} P(t) = \lim_{h\to\infty}\dfrac{200}{1+e^{5-t}} = \dfrac{200}{1} = 200$ students

(c) $P'(t) = \dfrac{d}{dt}200(1+e^{5-t})^{-1}$

$= 1200(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$= \dfrac{200e^{5-t}}{(1+e^{5-t})^2}$

$P''(t) = \dfrac{(1+e^{5-t})^2(200e^{5-t})(-1)-}{(1+e^{5-t})^4}$
$\dfrac{(200e^{5-t})(2)(1+e^{5-t})(e^{5-t})(-1)}{(1+e^{5-t})^4}$
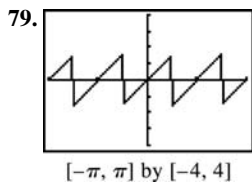
$= \dfrac{(1+e^{5-t})(-200e^{5-t})+400(e^{5-t})^2}{(1+e^{5-t})^3}$

$= \dfrac{(200e^{5-t})(e^{5-t}-1)}{(1+e^{5-t})^3}$

Since $P'' = 0$ when $t = 5$, the critical point of $y = P'(t)$ occurs at $t = 5$. To confirm that this corresponds to the maximum value of $P'(t)$, note that $P''(t) > 0$ for $t < 5$ and $P''(t) < 0$ for $t > 5$. The maximum rate occurs at $t = 5$, and this rate is

$P'(5) = \dfrac{200e^0}{(1+e^0)^2} + \dfrac{200}{2^2} = 50$ students per day.

Note: This problem can also be solved graphically.

79.

$[-\pi, \pi]$ by $[-4, 4]$

(a) $x \neq k\dfrac{\pi}{4}$, where k is an odd integer

(b) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) Where it's not defined, at $x = k\dfrac{\pi}{4}$, k an odd integer

(d) It has period $\dfrac{\pi}{2}$ and continues to repeat the pattern seen in this window.

80. Use implicit differentiation.

$x^2 - y^2 = 1$

$\dfrac{d}{dx}(x^2) - \dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(1)$

$2x - 2yy' = 0$

$y' = \dfrac{2x}{2y} = \dfrac{x}{y}$

$y'' = \dfrac{d}{dx}\dfrac{x}{y}$

$= \dfrac{(y)(1)-(x)(y')}{y^2}$

$= \dfrac{y - x\left(\dfrac{x}{y}\right)}{y^2}$

$= \dfrac{y^2 - x^2}{y^3}$

$= -\dfrac{1}{y^3}$

(since the given equation is $x^2 - y^2 = 1$)

At $(2, \sqrt{3})$, $\dfrac{d^2 y}{dx^2} = -\dfrac{1}{y^3} = -\dfrac{1}{(\sqrt{3})^3} = -\dfrac{1}{3\sqrt{3}}$.

81. (a) $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(t^3 - 2t + 3)$

$v(t) = 3t^2 - 2$

(b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt} = (3t^2 - 2)$

$a(t) = 6t$

$v(t) = 3t^2 - 2 = 0$

$t^2 = \dfrac{2}{3}$

(c) $t = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$

$t^2 = \dfrac{2}{3}$

$t = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$

(d) $v(t) = 3t^2 - 2 < 0$

$3t^2 < 2$

$t < \dfrac{\sqrt{6}}{3}$, and $t > 0$

$0 < t < \dfrac{\sqrt{6}}{3}$