2.1 RATES OF CHANGE AND LIMITS

Notecards from Section 2.1: How to Evaluate a Limit, Properties of Limits, Definition of a Limit at $x = c$, When Limits fail, and Limits you should know.

Limits

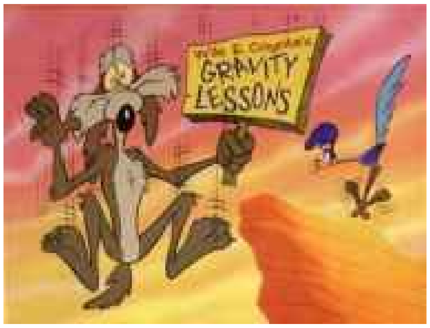
Limits are what separate Calculus from pre – calculus. Using a limit is also the foundational principle behind the two most important concepts in calculus, derivatives and integrals. Limits can be found using substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

Average and Instantaneous Velocity

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. (Speed is always positive … Velocity indicates direction and can be negative.) We are going to find the **average velocity**.

If an object is dropped from an initial height of h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, (in feet) of an object that has fallen for t seconds.

Example 1: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.



a) What is the position function for Wile E. Coyote?

b) Find Wile E.'s average velocity for the first 3 seconds.

c) Find Wile E.'s average velocity between $t = 2$ and $t = 3$ seconds.

d) Find Wile E.'s velocity at the instant $t = 3$ seconds.

The problem with part *d* is that we are trying to find the *instantaneous velocity*. Without the concept of a limit, we could not find the answer to part *d*. Using a limit to solve this problem involves studying what happens to the velocity as we get "close" to 3 seconds.

Example 2: Find the average velocity between $t = 2.5$ and $t = 3$ seconds.

Example 3: Find the average velocity between $t = 2.9$ and $t = 3$ seconds.

Example 4: Find the average velocity between $t = 2.99$ and $t = 3$ seconds.

Example 5: Find the average velocity between $t = 2.999$ and $t = 3$ seconds.

So, even though we cannot find the average velocity at exactly $t = 3$ seconds, we can discover what Wile E.'s velocity is approaching at $t = 3$ seconds.

Example 6: Use your calculator to generate a graph of $f(x)=\dfrac{x^2-4}{x-2}$; $x \neq 2$

a) Why is it that $x \neq 2$? What happens at $x = 2$?

b) Complete the table of values below to determine what happens as x gets "close" to 2.

x approaches 2 from the **left** ⟶ | ⟵ x approaches 2 from the **right**

x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Informal Definition of a Limit

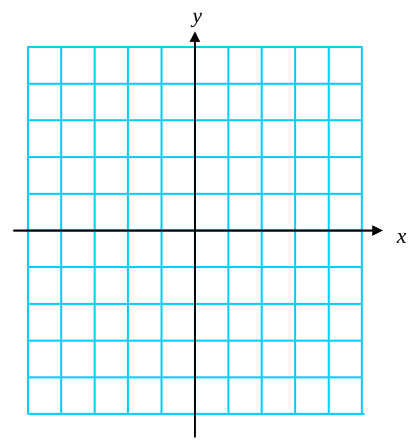
The limit of a function is the y-value (L) that the function approaches when the x-value gets closer and closer to a point c (from the left AND the right sides). We then say that the **limit of $f(x)$ as x approaches c is L**, and we write

$$\lim_{x \to c} f(x) = L.$$

c) Apply this definition to the function from above to find the $\lim_{x \to 2} f(x)$.

Example 7: Use the graph to find $\lim_{x \to 2} g(x)$, where g is defined as

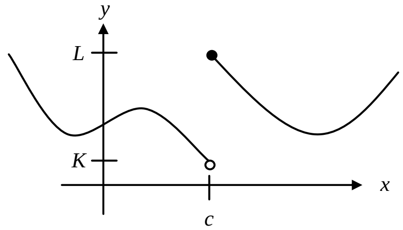
$$g(x)=\begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$



So, limits exist when the y-value gets close to a specific point (even if that point isn't actually part of the graph).

One – Sided Limits

Suppose we have the graph of $f(x)$ below. Notice … the function above does not approach the same y-value as x approaches c from the left and right sides. (This means that the $\lim_{x \to c} f(x)$ does not exist … more on this in a moment).



Sometimes, however, we are only interested in what the function approaches as x approaches from the right or left of c. We can say this using the following notation:

$\lim_{x \to c^+} f(x) = L$ … "the limit of $f(x)$ as x approaches c from the right is L"

$\lim_{x \to c^-} f(x) = K$ … "the limit of $f(x)$ as x approaches c from the left is K"

Thus, we can say that the limit of a function as x approaches any number c exists <u>if and only if</u> the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation we have

$$\boxed{\lim_{x \to c} f(x) \text{ exists} \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)}$$

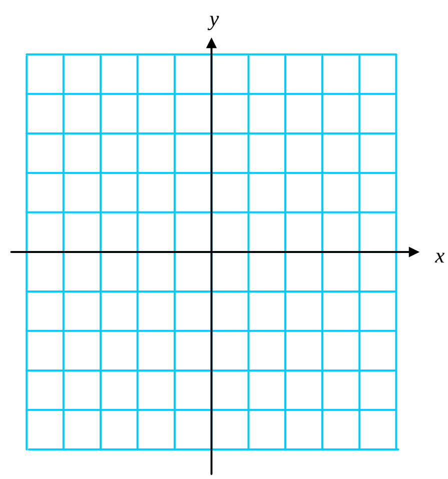
Example 8: Let $f(x) = \begin{cases} 5 - 2x & x > 2 \\ 3x + 1 & x \le 2 \end{cases}$

a) Graph $f(x)$

b) Find $\lim_{x \to 2^+} f(x)$.

c) Find $\lim_{x \to 2^-} f(x)$

d) What can you say about $\lim_{x \to 2} f(x)$?



When Limits Do Not Exist

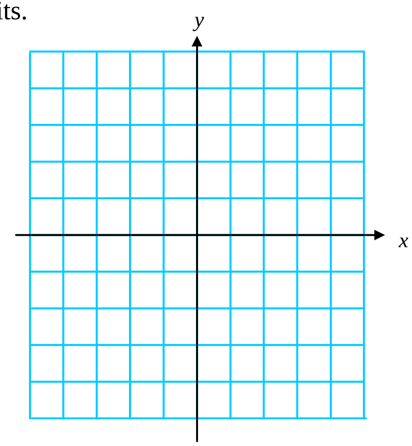
If there does not exist a number L satisfying the condition in the definition, then we say the $\lim\limits_{x \to c} f(x)$ does not exist.

Limits typically fail for three reasons:

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

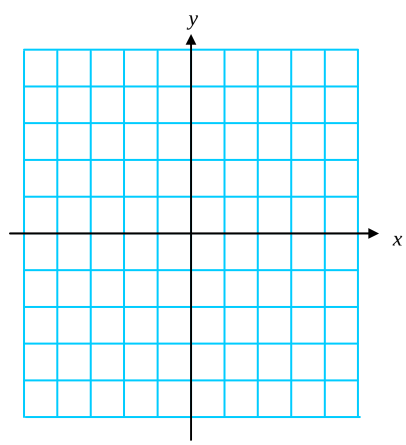
Example 9: Investigate (use a graph and/or table) the existence of the following limits.

(a) $\lim\limits_{x \to 0} \frac{|x|}{x}$



X	–0.5	–0.25	–0.1	–.01	–.001	0	.001	.01	.1	.25	.5
$f(x)$											

(b) $\lim\limits_{x \to 1} \frac{1}{(x-1)^2}$



x	0	.5	.9	.99	.999	1	1.001	1.01	1.1	1.5	2
$f(x)$											

(c) $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ … Try graphing this on your calculator.

First convince yourself that as you move to the right in the chart below x is actually getting closer and closer to 0.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$								

<u>Properties of Limits … Finding Limits By Direct Substitution</u>

For many "well – behaved" functions, evaluating the limit can be found by <u>direct substitution</u>. That is,

$$\lim_{x \to c} f(x) = f(c)$$

Such *well – behaved* functions are **continuous at *c***. We will study continuity of a function in §2.3.
The following theorems describe limits that can be evaluated by direct substitution.

Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

$\lim_{x \to c} b = b$

$\lim_{x \to c} x = c$

$\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$

$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = LK$

$\lim_{x \to c} \left[b \cdot f(x) \right] = bL$

$\lim_{x \to c} \left[f(x) \right]^{r/s} = L^{r/s}$
provided *r* and *s* are integers and $s \neq 0$

$\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$; provided $K \neq 0$

Example 10: Use the given information to evaluate the limits: $\lim_{x \to c} f(x) = 2$ and $\lim_{x \to c} g(x) = 3$

a) $\lim_{x \to c} \left[5g(x) \right]$

b) $\lim_{x \to c} \left[f(x) + g(x) \right]$

c) $\lim_{x \to c} \left[f(x) g(x) \right]$

d) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$

When dealing with a constant value for *c*, realize that the properties in the box above basically allow us to evaluate a limit by plugging in the value of *c* everywhere there is an *x*. Be careful with your variables.

Example 11: Find each limit

a) $\lim_{x \to 1} \left(-x^2 + 1 \right)$

b) $\lim_{x \to 3} \dfrac{\sqrt{x+1}}{x-4}$

c) $\lim_{h \to 0} \left(3h^2 + 2h \right)$

d) $\lim_{h \to 0} \left(3x^2 - 2xh + 5h \right)$

Other Strategies for Finding Limits

If a limit cannot be found using direct substitution, then we will use other techniques to evaluate the limit.

♬: Keep in mind that some functions do not have limits.

If direct substitution yields the meaningless result $\frac{0}{0}$, then you ***cannot determine*** the limit in this form.

The expression that yields this result is called an **Indeterminate Form**. … *DO SOMETHING ELSE!*
When you encounter this form, you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *cancel like factors*, and a second way is to *rationalize the numerator*, and a third is to simplify the algebraic expression and evaluate the limit by direct substitution again.

Example 12: Find the limit: $\lim_{x\to -1} \frac{2x^2 - x - 3}{x+1}$

Example 13: Find the limit (if it exists): $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

Example 14: Find the limit (if it exists): $\lim_{x\to 0} \frac{\left[1/(x+4)\right]-(1/4)}{x}$

Example 15: Investigate the $\lim_{x \to 0} \frac{\sin x}{x}$ by sketching a graph and making a table.

You must understand that while using a graph and/or a table, we may be able to determine what a limit is, we have not proved it until we algebraically confirm the limit is what we think it is. The proof (see page 2-7) of the above limit requires the use of the sandwich theorem. (see below) … YOU SHOULD REMEMBER THIS LIMIT!

Example 16: Evaluate the following limits, showing all your work where appropriate.

a) $\lim_{☺ \to 0} \frac{\sin ☺}{☺}$

b) $\lim_{x \to 0} \frac{\sin 5x}{4x}$

c) $\lim_{x \to 0} \frac{\sin x}{5x^2 + x}$

The Sandwich Theorem (a.k.a. The Squeeze Theorem)

The Sandwich Theorem

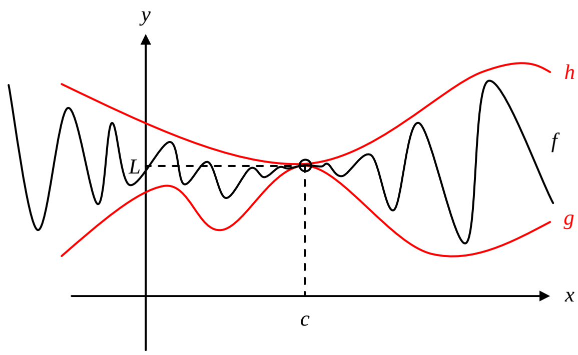
If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c*, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

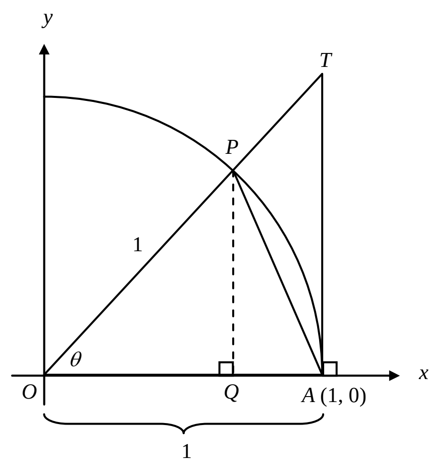
$$\lim_{x \to c} f(x) = L.$$

In other words, if we "sandwich" the function f between two other functions *g* and *h* that both have the same limit as *x* approaches *c*, then f is "forced" to have the same limit too.



Example 17: Prove that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. To do this, we are going to use the figure below. Admittedly, the toughest part of using the sandwich theorem is finding two functions to use as "bread" ☺.

First, we need to find $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$. In order to do this we need to restrict θ so that $0 < \theta < \frac{\pi}{2}$. Why are we able to do this?



a) Find the area of $\triangle OAP$.

b) Find the area of sector *OAP*.

c) Find the area of $\triangle OAT$.

d) Set up an inequality with the three areas from parts *a*, *b*, and *c*.

e) Divide all three parts by $\frac{1}{2}\sin\theta$. Why do the inequality signs stay the same?

f) Make the middle term $\frac{\sin\theta}{\theta}$. *Hint*: If your middle term doesn't look anything like this, start over! ☺

g) Use the Sandwich Theorem to show that $\lim_{x \to 0^+} \frac{\sin\theta}{\theta} = 1$.

h) Show that $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function.

i) Since $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function, what can you conclude about $\lim_{x \to 0^-} \frac{\sin\theta}{\theta}$?

j) Explain why we can conclude that $\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$.

2.2 LIMITS INVOLVING INFINITY

Notecards from Section 2.2: Definition of a horizontal asymptote, Definition of a vertical asymptote, End Behavior, End Behavior Models, Oblique (Slant) Asympotes

We are going to look at two kinds of limits involving infinity. We are interested in determining what happens to a function as x approaches infinity (in both the positive and negative directions), and we are also interested in studying the behavior of a function that approaches infinity (in both the positive and negative directions) as x approaches a given value.

Finite Limits as $x \to \pm\infty$

Example 1: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x)=\frac{1}{x}$.

Example 2: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x)=\frac{2x-1}{x+3}$.

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b$$

Rational Functions (two polynomial functions divided) have the same horizontal asymptote in both directions.

Horizontal Asymptotes of Rational Functions

Definition: End Behavior Model

For a ***rational*** function $\frac{ax^m+\cdots}{bx^n+\cdots}$, where m is the degree of the numerator and n is the degree of the denominator. The end behavior model can be written as $\frac{ax^m}{bx^n}$ or $\frac{a}{b}x^{m-n}$.

We can use the end behavior models of *rational* functions to identify any *horizontal asymptotes* the function.

If $f(x) = ax^m + \cdots$ and $g(x) = bx^n + \cdots$ then $\dfrac{f(x)}{g(x)}$ takes on three different forms.

	End Behavior Model	End Behavior	Asymptote
Degrees are equal $m = n$			
Larger degree in denominator $m < n$			
Larger degree in numerator $m > n$			

Example 3: For each example below do the following:

i) Write the end behavior model.
ii) Evaluate each limit.
iii) Determine whether or not there are any horizontal asymptotes. If so, what is the equation?
iv) Determine whether or not there are any slant (oblique) asymptotes. If so, what is the equation?

a) $\lim\limits_{x\to\infty} \dfrac{2x+5}{3x^2-6x+1}$

b) $\lim\limits_{x\to\infty} \dfrac{2x^2-3x+5}{x^2+1}$

c) $\lim\limits_{x\to\infty} \dfrac{x^4+x^3+9}{3x-3}$

Horizontal Asymptotes of Non-Rational Functions

In the previous examples, if there was a limit as x approached positive infinity, the limit as x approached negative infinity was the same. This occurs whenever you have a *rational function*.

A general rule for functions that are divided (not necessarily rational functions) is that if the denominator "grows" faster than the numerator, the limit as x approaches infinity will be 0. If the numerator "grows" faster than the denominator, then as x approaches infinity, the limit will not exist.

Example 4: In the last section we proved that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. Investigate $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$.

There are also functions that have more than one horizontal asymptote. [Recall the definition of a horizontal asymptote]

Example 5: Investigate $\lim_{x\to\infty}\frac{5+2^x}{3-2^x}$ and $\lim_{x\to-\infty}\frac{5+2^x}{3-2^x}$

Infinite Limits as $x \to a$

A second type of limit involving infinity is to determine the behavior of the function as x approaches a certain value when the function increases or decreases without bound.

Example 6: Investigate $\lim_{x\to 0^+}\frac{1}{x}$ and $\lim_{x\to 0^-}\frac{1}{x}$.

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

***This occurs whenever there is a value of x that gives you a 0 in the denominator (but not the numerator).

Important ♬ : Infinity is NOT a number, and thus the limit FAILS to exist in both of these cases. If this seems confusing, then use the notation as $x \to a$ (from the right or left), then the function $f(x) \to \pm\infty$.

Example 7: Find the vertical asymptotes of each function, and describe the function's behavior of as x approaches from the left and right of each asymptote.

a) $f(x) = \frac{x^2-1}{2x+4}$

b) $h(x) = \frac{1-x}{2x^2-5x-3}$

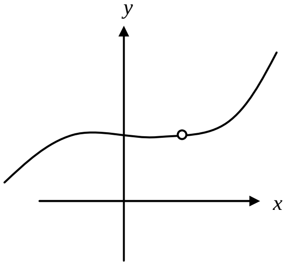
c) $B(x) = \frac{x-2}{3x^2-5x-2}$

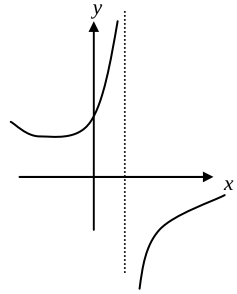
2.3 CONTINUITY

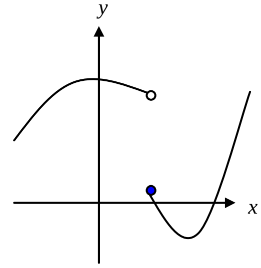
Notecards from Section 2.3: Definition of continuity at $x = c$, Types of Discontinuities, Intermediate Value Theorem

In §2.1 we referred to "well behaved" functions. "Well behaved" functions allowed us to find the limit by direct substitution. "Well behaved" functions turn out to be continuous functions. In this section we will discuss continuity at a point, and the different types of discontinuities.

In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil". THIS IS NOT A DEFINITION YOU SHOULD USE AFTER TODAY!!! The following graphs demonstrate three types of discontinuous graphs.







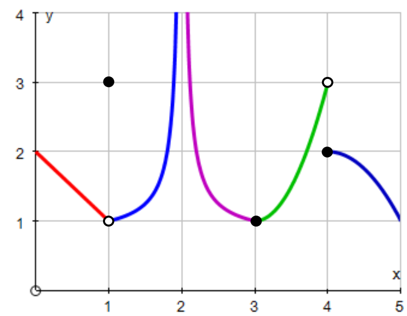
<u>Discontinuities: Removable versus Non-Removable</u>

To say a function is discontinuous is not sufficient. We would like to know what type of discontinuity exists. If the function is not continuous, but I <u>could make it continuous by appropriately defining or redefining $f(c)$,</u> then we say that f has a **removable discontinuity**. Otherwise, we say f has a **non-removable discontinuity**.

Once again, informally we say that f has a <u>removable discontinuity</u> if there is a "<u>hole</u>" in the function, but f has a non-removable discontinuity if there is a "jump" or a vertical asymptote.

Example 1: Which (if any) of the three graphs above have a removable discontinuity?

Example 2: Find the points (intervals) at which the function below is continuous, and the points at which it is discontinuous.



<u>Definition</u>: Continuity at a Point
A function $y = f(x)$ is **continuous point *c*** if $$\lim_{x \to c} f(x) = f(c)$$
♬: This last statement implies that $\lim_{x \to c} f(x)$ exists … which is true IF AND ONLY IF the left and right limits are equal! It also implies that the function value at c … $f(c)$ exists.

Especially useful for piecewise functions, the following is another way to view the definition of continuity …

$$\boxed{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)}$$

Example 3: Go back to the last picture. For c = 1, 2, and 4, find $f(c)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$, and $\lim_{x\to c} f(x)$ if they exist. (Can you see how all the parts of the definition of continuity are important?)

a) for $c = 1$ b) for $c = 2$ c) for $c = 4$

You don't always get a picture, so you're going to have to do this algebraically as well.

Example 4: Determine whether each function is continuous or not. If it is not continuous, use the definition of continuity to explain why.

(a) $f(x) = \dfrac{1}{x-1}$ (b) $g(x) = \dfrac{2x^2 + x - 6}{x+2}$

(c) $h(x) = \begin{cases} -2x+3 & ; x < 1 \\ x^2 & ; x \ge 1 \end{cases}$

Example 5: Use the definition of continuity to find the value of a so that $g(x)$ will be continuous for all real numbers.

$$g(x) = \begin{cases} x^2 + 7 & \text{if } x \ge 1 \\ x + a & \text{if } x < 1 \end{cases}$$

Example 6: Use the definition of continuity to find the value of k so that $h(x)$ will be continuous for all real numbers.

$$h(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1} & \text{if } x \ne 1 \\ k & \text{if } x = 1 \end{cases}$$

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Constant multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$; $g(c) \neq 0$

The Intermediate Value Theorem (IVT)

If f is continuous on the closed interval [a, b] then f takes on every value between $f(a)$ and $f(b)$.

Suppose k is any number between $f(a)$ and $f(b)$, then there is at least one number c in [a, b] such that $f(c) = k$.

♬: The Intermediate value theorem tells you that at least one c exists, but it does not give you a method for finding c. This theorem is an example of an *existence theorem*.

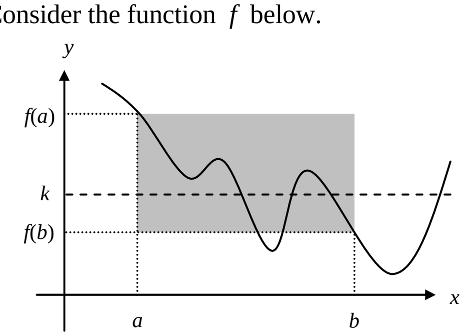
Example 7: In the Intermediate Value Theorem …

a) What are the necessary requirements in order to apply this theorem?

b) k is on which axis?

c) c is on which axis?

Example 8: Consider the function f below.



- Is f continuous on [a, b] ?
- Is k between $f(a)$ and $f(b)$?
- In this example, if $a < c < b$, then there are ________ c's such that $f(c) = k$.
- Label the c's on the graph as $c_1, c_2, \ldots$

Example 9: Let $f(x) = \frac{x^2 + x}{x - 1}$. Verify that the Intermediate Value Theorem applies to the interval $\left[\frac{5}{2}, 4\right]$ and explain why the IVT guarantees an x-value of c where $f(c) = 6$.

2.4 RATES OF CHANGE AND TANGENT LINES

Notecards from Section 2.4: Definition of Average Rate of Change, Definition of Instantaneous Rate of Change.

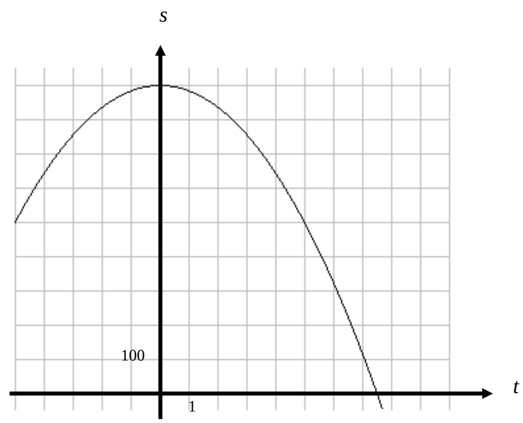
Average Rates of Change

Example 1: Remember this example? … Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

Let's look at this problem from a graphical perspective. The equation that models Wile E.'s height at any time t is given by

$$s(t) = -16t^2 + 900$$

A graph of this equation is shown below.



a) Find the points on the graph that correspond to Wile E's position at $t = 0$ and $t = 5$ seconds.

b) Draw the line that passes through these points, and find the slope. What does this value mean?

The line you drew on the graph can be called a *secant line*. <u>A secant line is a line through any two points on a curve</u>. Just like we did in this example, we can always think of the **average rate of change** as the slope of the secant line. To find the slope of the secant line above we divided the total change in s by the total change in t. To find the average rate of change in the position (a.k.a. velocity) we found the total change in position divided by the total change in time.

We are able to find Wile E.'s average velocity for any period of time following the same procedure as above. Do you remember the problem we had finding the velocity of poor Wile E. Coyote at an exact moment in time? If we wanted to find the velocity of Wile E. Coyote at <u>*exactly*</u> 5 seconds, we tried to determine the average velocity using values of t that were closer and closer to $t = 5$.

c) Find Wile E.'s average velocity (rate of change) from $t = 4$ to $t = 5$ seconds. Graphically show this above.

d) Find Wile E.'s average velocity (rate of change) from $t = 4.5$ to $t = 5$ seconds. Graphically show this above.

e) Find Wile E.'s average velocity (rate of change) from $t = 4.9$ to $t = 5$ seconds. Graphically show this above.

d) What do you think would be the graphical interpretation of the velocity at exactly 5 seconds?

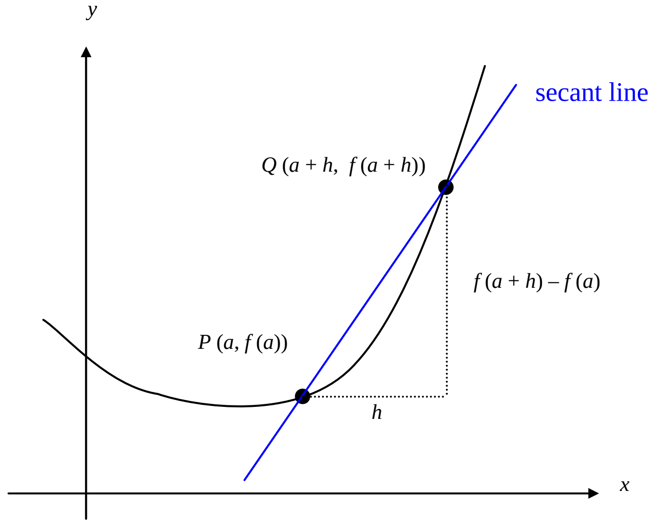
<u>Slope of a Tangent Line … aka "Slope of a Curve at a Point"</u>

For a general curve, the equation of the tangent line simply boils down to finding the *<u>slope</u>* of the tangent line. (Since we already have a <u>point</u> of tangency, if we knew the slope we would be able to write the equation of a line.)

We can *approximate* the slope of the tangent line using a **secant line**.

If $P\ (a,\ f(a))$ is the point of tangency we are concerned with, then we can pick an arbitrary point Q on the graph and estimate the tangent line at P using the slope of the secant line through P and Q.

Example 2: What is the slope of the secant line?



The beauty of this procedure is that you can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency.

Example 3: How do we get closer and closer to the point of tangency?

Draw at least 3 more secant lines, using a point closer to P each time.

As $h \to 0$, the slope of the secant line approaches the slope of the tangent line.

<u>Slope of a Curve</u>

The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the "track" were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point.

Using the slope of the secant line from the last example, we have the following definition.

<u>Slope of a Curve at a Point</u>

The **slope of the curve** $y = f(x)$ at the point $P\ (a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

♬: The expression $\dfrac{f(a+h) - f(a)}{h}$ is called a **difference quotient**.

Your goal is to SIMPLIFY the difference quotient, THEN evaluate the limit as h approaches 0.

<u>The difference quotient is simplified when you have cancelled the *h* in the denominator in an algebraically correct way.</u>

Finding the Equation of a Tangent Line

The **tangent line to the curve** at P is the equation of a line. Thus we need a point … P … and a slope … the limit as h goes to zero of the difference quotient.

The slope might not exist because the limit doesn't exist (discontinuity, vertical asymptote, oscillating function) or because the tangent line has a vertical slope.

** For a tangent line that exists, but has no slope, this definition doesn't quite fit. To cover the possibility of a vertical tangent line, we can use the following definition.

If f is continuous at a and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\pm\infty$$

the vertical line, $x = a$ is a **vertical tangent line** to the graph of f.

Example 4: What types of graphs would have vertical tangent lines?

Example 5: Before we use this definition, be sure to become comfortable with the notation $f(a + h)$.

a) If $f(x)=\frac{1}{x}$, what is $f(a)$? … $f(a+h)$?

b) If $f(x)=x^2-4x$, what is $f(a)$? … $f(a+h)$?

c) If $f(x)=\sqrt{4x+1}$, what is $f(a)$? … $f(a+h)$?

Example 6: Back to example 1 … Let $f(x)=-16x^2+900$. Find the slope of the curve at $x = 5$.

♬ : This would be the **instantaneous velocity** of Wile E. Coyote at exactly 5 seconds!

Normal Line

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

Example 7: Let $f(x) = \dfrac{1}{x+1}$.

a) Find the slope of the curve at $x = a$.

b) Find the slope of the curve at $x = 2$.

c) Write the equation of the tangent line to the curve at $x = 2$.

d) Write the equation of the normal line to the curve at $x = 2$.