3.1 DERIVATIVE OF A FUNCTION

Notecards from Section 3.1: Definition of a Derivative (2 of the 3 ways), Definition of the existence of a derivative at $x = c$ and at an endpoint.

In the last chapter we used a limit to find the slope of a tangent line. Without knowing it, you were finding a derivative all along. A derivative of a function is one of the two main concepts from calculus. The other is called an integral, and we will not get to that until later. The only change from the limit definition we used before, is that we are going to treat the derivative as a function derived from *f*.

Definition of a Derivative

The **derivative** of a function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Anywhere that the derivative exists, we say that the function is *differentiable*.

Thus the derivative is a function that gives the slope of the function at any point.

Example 1: Other notation used to denote the derivative (we will use most of these). REMEMBER THESE!!!!

Example 2: Use the definition of the derivative to find $f'(x)$.

a) $f(x) = 3x + 2$

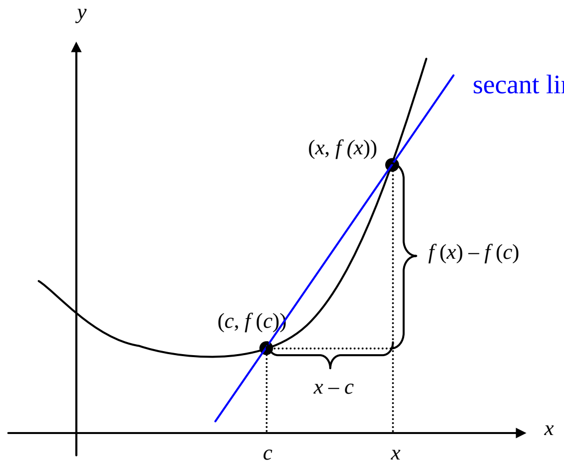
b) $f(x) = x^3 + x^2$

Alternative Definition #1 of the Derivative

An alternative definition of the derivative of *f* at *c* is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.



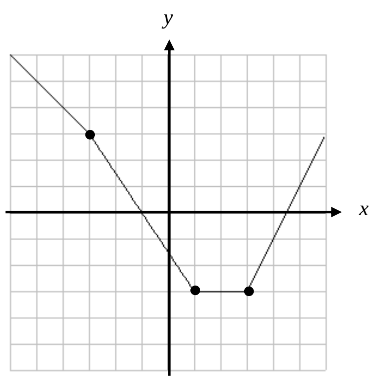
♬: What this alternative definition allows us to do is to examine the behavior of a function as x approaches c from the left or the right. The limit exists (and thus the derivative) as long as the left and right limits exist and are equal.

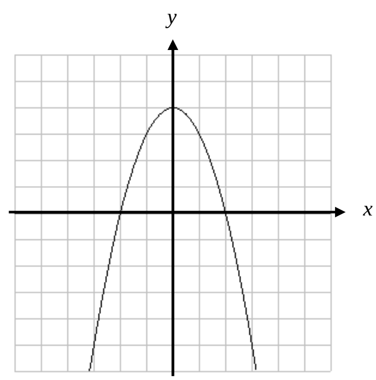
Example 3: Use the alternative definition to find the derivative of $f(x) = \frac{1}{\sqrt{x}}$.

Relationships between the Graphs of f and f'.

Since a derivative at any point is equivalent to the slope of the function at that point, we can *estimate* what the original function looks like when we are given the graph of the derivative and vice – versa.

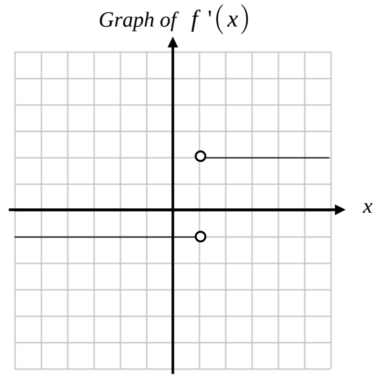
Example 4: Given the graph of *f*, sketch the graph of the derivative on the same set of axes.



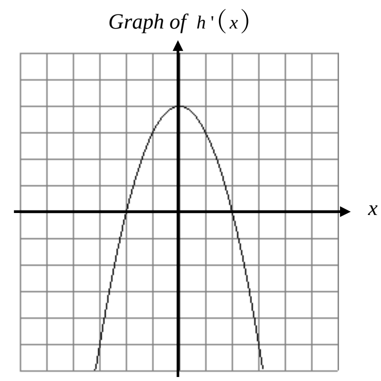


For a few other examples … see http://people.hofstra.edu/stefan_waner/Realworld/calctopic1/derivgraph.html

Example 5: Given the graph of f', sketch the graph of the function f on the same set of axes if you know that $f(0) = 2$. Why is it necessary to know this last part?



Example 6: Suppose the graph below is the graph of the derivative of *h*.



a) What is the value of $h'(0)$? What does this tell us about $h(x)$?

b) Using the graph of $h'(x)$, how can we determine when the graph of $h(x)$ is going up? How about going down?

c) The graph of $h'(x)$ crosses the x-axis at $x = 2$ and $x = -2$. Describe the behavior of the graph of $h(x)$ at these points.

3.2 DIFFERENTIABILITY

Notecards from Section 3.2: Where does a derivative NOT exist, Definition of a derivative (3^{rd} way).

The focus on this section is to determine when a function fails to have a derivative. For all you non-English majors, the word *differentiable* means you are able to take a derivative, or the derivative exists.

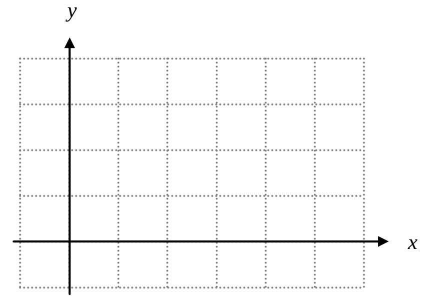
Example 1: Using the grid provided, graph the function $f(x) = |x-3|$.

a) What is $f'(x)$ as $x \to 3^-$?

b) What is $f'(x)$ as $x \to 3^+$?

c) Is f continuous at $x = 3$?

d) Is f differentiable at $x = 3$?

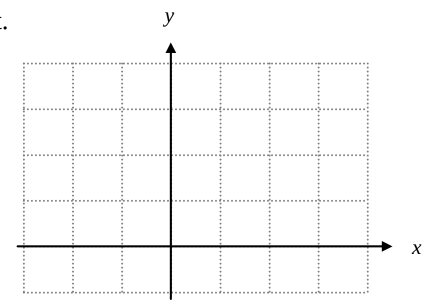


Example 2: Graph $f(x) = x^{2/3}$

a) Describe the derivative of $f(x)$ as x approaches 0 from the left and the right.

b) Suppose you found $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$.

Using this formula, what is the value of the derivative when $x = 0$?

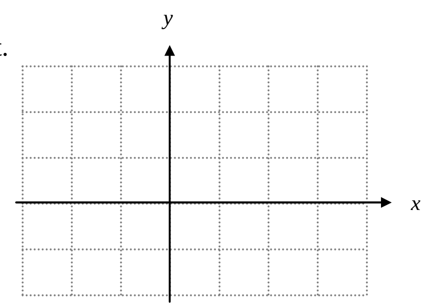


Example 3: Graph $f(x) = \sqrt[3]{x}$

a) Describe the derivative of $f(x)$ as x approaches 0 from the left and the right.

b) Suppose you found $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$.

Using this formula, what is the value of the derivative when $x = 0$?



These last three examples (along with any graph that is not continuous) are *NOT differentiable*. The first graph had a "*corner*" or a sharp turn and the derivatives from the left and right did not match. The second graph had a "*cusp*" where secant line slope approach positive infinity from one side and negative infinity from the other. The third graph had a "*vertical tangent line*" where the secant line slopes approach positive or negative infinity from both sides.

The first two can be referred to as a "pointy place" .

In all three of the previous examples the functions were continuous, but failed to be differentiable at certain points.

Continuity does not guarantee differentiability, but it does work the other way around.

> **Differentiability Implies Continuity**
>
> If f is differentiable at $x = c$, then f is continuous at $x = c$.

Example 4: For the logical statement … if A, then B … the converse is written … if B, then A.
The converse of the statement in the box is NOT true! What is the converse?

Example 5: The contrapositive of any statement is logically equivalent to the original statement. For the logical statement … if A, then B … the contrapositive is written … if not B, then not A. What is the Contrapositive to the statement in the box?

Example 6: If you are given that f is differentiable at $x = 2$, then explain why each statement below is true?

a. $\lim_{x \to 2} f(x)$ exists

b. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ exists.

c. $f(2)$ exists

d. $\lim_{x \to 2} \frac{f(x) - f(2)}{x-2}$ exists.

Example 7: If f is a function such that $\lim_{x \to -3} \frac{f(x) - f(-3)}{x+3} = 2$, which of the following must be true?

A) The limit of $f(x)$ as x approaches –3 does not exist.

B) f is not defined at $x = -3$.

C) The derivative of f at $x = -3$ is 2.

D) f is continuous at $x = 2$.

E) $f(-3) = 2$

Using the TI – 83+

Most graphing calculators can take derivatives at certain points. In fact, it is necessary on the AP exam that you have a calculator that will take the derivative at a given point. However, they use a different method of calculating the derivative than our earlier definitions.

♬: The TI-89 will actually find the derivative formula, but when a derivative is needed on the calculator portion it has been asked only to evaluate the derivative at a point, thus removing the advantage of having a TI-89 over a TI-83 (or 84).

Example 8: We used the following formula to find the derivative in the last section. Provide a geometric interpretation (picture) of this formula:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 9: Here's a <u>third way to define the derivative</u>. Draw a picture to represent this formula:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

Your graphing calculator uses the concept of this last definition to calculate derivatives.

To use your graphics calculator to find the derivative, use the `nDeriv(` function on the TI-83+. To access this function press [MATH], then 8 (or use [▲] and [▼] to go to `nDeriv(` and press [ENTER]) . The `nDeriv(` function works as follows:

`nDeriv(function,variable,value)`

Where "*function*" is the function you want to find the derivative of, "*variable*" is the variable you are differentiating with respect to (usually *x*), and "*value*" is the point at which you want to find the derivative.

♬: Many times it is easier to type the function into `Y1`, and then enter `nDeriv(Y1,x,#)`.

Example 10: Use your calculator to find the derivative of $f(x) = x^2 - 3x + 2$ at $x = -3$. Express your answer with the correct notation.

Example 11: Use your calculator to find the derivative of the three examples at the beginning. What problems do you find? Why?

3.3 RULES FOR DIFFERENTIATION

Notecards from Section 3.3: Power Rule; Product Rule; Quotient Rule

Drum Roll please … [In a Deep Announcer Voice] … And now … the moment YOU'VE ALL been waiting for …

Rule #1 Derivative of a Constant Function

If c is any constant value, then $\frac{d}{dx}[c]=0$

This should not be too earth shattering to you, since the slope of a constant function is always 0!

Example 1: Let $f(x)=5$. Find $f'(x)$.

Rule #2 Power Rule

If n is any number, then $\frac{d}{dx}[x^n]=n\cdot x^{n-1}$, provided x^{n-1} exists.

♬: In section 3.3 your book distinguishes between n being a positive integer (rule 2), n being a negative integer (rule 7) and n being a rational number (rule 9, section 3.7). The distinction is made so that they may prove each separate case in the book. However, the use of the power rule is unchanged for all three different values of n.

The KEY to using the power rule is to get comfortable using exponent rules to write a function as a power of x.

Example 2: Let $f(x)=x^5$. Find $f'(x)$.

Example 3: Let $f(x)=\sqrt[3]{x^2}$. Find $f'(x)$.

Example 4: Let $f(x)=\frac{1}{x^4}$. Find $f'(x)$.

Rule 3: The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}[cu]=c\frac{du}{dx}.$$

Example 5: Let $y=5x^7$. Find $\frac{dy}{dx}$.

Example 6: Let $g(x)=\frac{4}{5x^3}$. Find $g'(x)$.

Rule 4: *The Sum and Difference Rule*

If *u* and *v* are differentiable functions of *x*, then wherever *u* and *v* are differentiable

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$

Example 7: Let $y = x^3 + 4x^2 - 2x + 7$. Find y'.

Example 8: Let $g(x) = \frac{3}{(-2x)^4} - \frac{x}{2} + \frac{1}{4}$. Find $g'(x)$.

Example 9: Find the equation of the tangent line to the function $f(x) = 4x^3 - 6x + 5$ when $x = 2$.

Example 10: Let $h(x) = (x^2 + 1)(2x - 5)$. Find $h'(x)$.

Example 11: The volume of a cube with sides of length *s* is given by $V = s^3$. Find $\frac{dV}{ds}$ when $s = 4$ centimeters.

Using Rule 4, we know that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This *does not work for the product and quotient of two functions*. To illustrate this, we look at the following example.

Example 12: Find $\frac{d}{dx}\left[x^2 \cdot 3x\right]$.

Rule 5: *The Product Rule*

If *u* and *v* are differentiable functions of *x*, then

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

This is also written as

$$\frac{d}{dx}[uv] = uv' + vu'$$

For polynomial functions it is not always necessary to use the product rule, however, with trigonometric, exponential, logarithmic, and other functions, it is a necessary tool.

Example 13: Let $y=\left(3+2\sqrt{x}\right)\left(5x^3-7\right)$. Find $\frac{dy}{dx}$ first without using the product rule, then with the product rule.

<u>*Rule 6*</u>: *The Quotient Rule*

If *u* and *v* are differentiable functions of *x*, and $v\neq 0$

$$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

This is also written as

$$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}.$$

Looks GREAT! Doesn't it?! Well, luckily for you, it ain't that bad. With thanks to Snow White and the Seven Dwarfs, if we replace *u* with *hi* and *v* with *ho* (*hi* for high up there in the numerator and *ho* for low down there in the denominator), and letting *D* stand for "the derivative of", the formula becomes

$$D\left(\frac{hi}{ho}\right)=\frac{ho\,D(hi)-hi\;D(ho)}{(ho)^2}$$

In words, that is "**<u>ho dee hi minus hi dee ho over ho ho</u>**". Now, if Sleepy and Sneezy can remember that, it shouldn't be any problem for you.

Example 14: Find $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

Example 15: Find $\frac{d}{dx}\left[\frac{5x^2}{x^3+1}\right]$

Second and Higher Order Derivatives

The first derivative of *y* with respect to *x* is denoted y' or $\frac{dy}{dx}$. The second derivative of *y* with respect to *x* is denoted y'' or $\frac{d^2y}{dx^2}$. The second derivative is an example of a higher – order derivative. We can continue to take derivatives (as long as they exist) using the following notation:

First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n^{th} derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

Example 16: Find $\frac{d^4}{dx^4}\left[-5x^8+2x^6-9x^3+32x-1\right]$.

Example 17: Let $f(x)=\frac{x}{x-1}$. Find $f''(x)$.

Example 18: If $f^{(4)}(x)=2\sqrt{x}$, find $f^{(5)}(x)$.

3.4 VELOCITY AND OTHER RATES OF CHANGE

Notecards from Section 3.4: Relationships between Position, Velocity, and Acceleration; Velocity vs. Speed

Instantaneous Rates of Change

We have already seen that the instantaneous rate of change is the same as the slope of the tangent line and thus the derivative at that point. Unless we use the phrase “average rate of change”, we will assume that in calculus the phrase “rate of change” refers to the instantaneous rate of change.

Example 1: The length of a rectangle is given by $2t + 1$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time, and indicate the units of measure for this rate.

Motion along a line

When a spring attached to a wall is stretched and then released, it moves back and forth. This motion can be described using functions involving sine and cosine (which we are not ready for just yet …). However, motion along a line in other circumstances (either horizontal or vertical lines) can also be described using functions. Typically, we use a position function $s(t)$ to describe the position s of an object after t seconds.

Relationships between Position, Velocity, and Acceleration

The *displacement* of an object is the TOTAL CHANGE IN POSITION.

The *average velocity* of the object is described as TOTAL CHANGE IN POSITION (displacement) divided by the TOTAL CHANGE IN TIME. It can be thought of as the slope of the line connecting two points on a position function.

The *instantaneous velocity* of the object is the *derivative of the position function*. Unless term “average velocity” is used, we will assume velocity refers to instantaneous velocity. It is the slope of a tangent line to the position function.

Positive Velocity indicates movement in the positive direction.
Negative velocity indicates movement in the negative direction.

Speed is the absolute value of velocity. Thus speed is always positive value, whereas, ***velocity indicates direction***.

Acceleration is the rate of change in velocity, implying then that acceleration is the *derivative of velocity*. Since it is the derivative of velocity, it is also the *second derivative of position*.

Example 2: Bugs Bunny has been captured by Yosemite Sam and forced to “walk the plank”. Instead of waiting for Yosemite Sam to finish cutting the board from underneath him, Bugs finally decides just to jump. Bugs’ position, s, is given by $s(t) = -16t^2 + 16t + 320$, where s is measured in feet and t is measured in seconds.

a) What is Bugs’ displacement from $t = 1$ to $t = 2$ seconds?

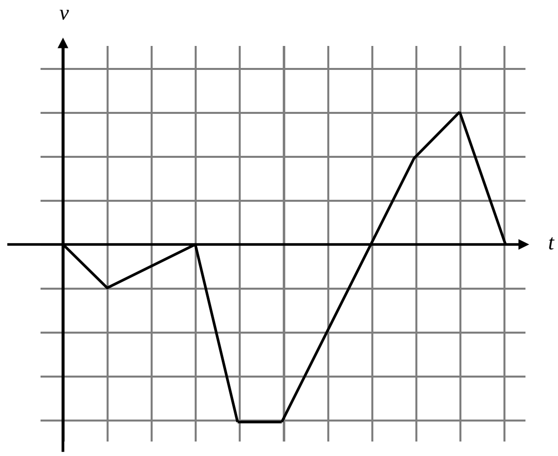
b) When will Bugs hit the ground?

c) What is Bugs’ velocity at impact? (What are the units of this value?)

d) What is Bugs’ speed at impact?

e) Find Bugs’ acceleration as a function of time. (What are the units of this value?)

Example 3: Suppose the graph below shows the velocity of a particle moving along the x – axis. Justify each response.



a) Which way does the particle move first?

b) When does the particle stop?

c) When does the particle change direction?

d) When is the particle moving left?

e) When is the particle moving right?

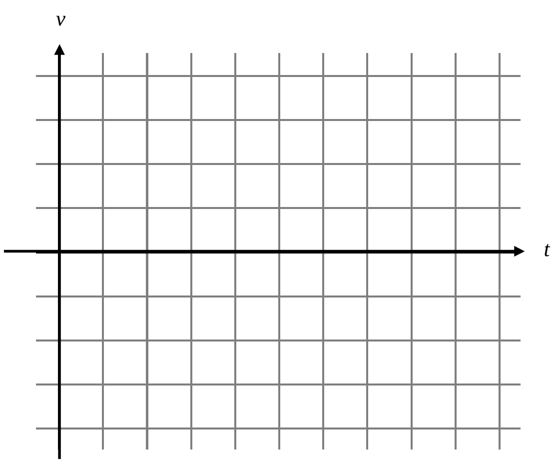
f) When is the particle speeding up?

g) When is the particle slowing down?

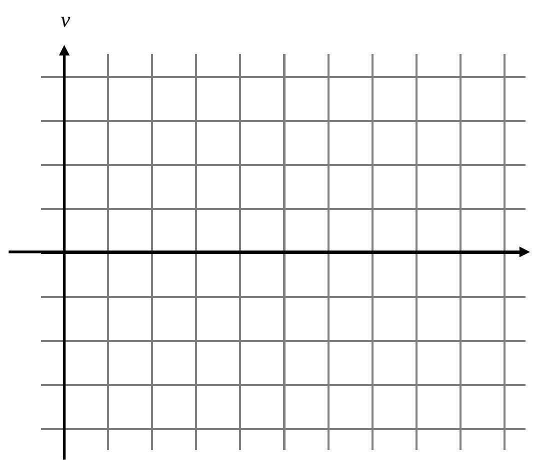
h) When is the particle moving the fastest?

i) When is the particle moving at a constant speed?

h) Graph the particle's acceleration for $0 < t < 10$.

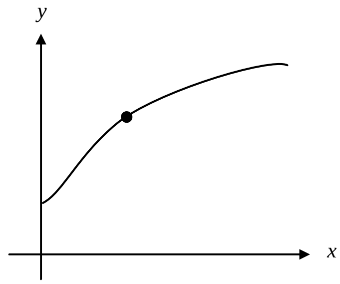


j) Graph the particle's speed for $0 \leq t \leq 10$.



Derivatives in Economics

Economists use calculus to determine the rate of change of costs with respect to certain factors. *Example*: Draw a tangent line at the indicated point on the function below.



Suppose the original function was a profit function. Can we use the tangent line you drew to estimate the profit?

Label the initial point (x, $P(x)$). If we increased x by 1, what is the actual change in the profit?

Now consider the tangent line …

The slope of the tangent line is given by ____________, and slope of a line is given by _________.

If we consider a change in x of 1, then the slope of the tangent line = the change in y …. ($P'(x)=\Delta y$).

This is the underlying principle in the following definitions. If x changes by 1 unit, then the change in y is approximately the value of the derivative at the original x.

Marginal Cost: The marginal cost at x, given by $C'(x)$, is the approximate cost of the (x + 1)st item.

Marginal Revenue: The marginal revenue at x, given by $R'(x)$, is the approximate revenue generated by the (x + 1)st item.

Marginal Profit: The marginal profit at x, given by $P'(x)$, is the approximate profit generated by the (x + 1)st item.

Example 4: Suppose that the daily cost, in dollars, of producing x radios is $C(x)=0.002x^3+0.1x^2+42x+300$, and currently there are 40 radios produced daily.

a) What is the current daily cost?

b) What would the actual additional daily cost of increasing production to 41 radios daily?

c) What is the marginal cost of the 41st unit?

Example 5: Suppose the cost of producing x units is given by $c(x)=4x^2+\dfrac{300}{x}$. What is the marginal cost of producing the 11th unit?

3.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Notecards from Section 3.5: Derivatives of 6 Trig Functions

The goal for this lesson is to introduce the derivatives of the 6 trigonometric functions. It is vital to your success in this course that you commit all 6 of these to memory *as quickly as possible*!

Using the derivatives of sin *x* and cos *x*, you can find the derivatives of the other 3 trig functions as well. The remaining portion of these notes will show how to use the definition of a derivative to find the derivatives of the sine function, followed by a few examples using these 6 rules along with the power, product, and quotient rules.

Derivatives of the Six Basic Trigonometric Functions

$\frac{d}{dx}[\sin x]=\cos x$	$\frac{d}{dx}[\cos x]=-\sin x$	$\frac{d}{dx}[\tan x]=\sec^2 x$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\frac{d}{dx}[\csc x]=-\csc x\cot x$

First, to prove the derivative of sine, we need a few background bits of information …

You should already know this limit … $\lim_{x\to 0}\frac{\sin x}{x}$ = ____.

Example 1: Investigate $\lim_{x\to 0}\frac{\cos x-1}{x}$.

Example 2: To prove this algebraically, multiply the top and bottom by $(\cos x+1)$, then evaluate the limit.

Example 3: Find the derivative of sin *x* using the limit definition of the derivative.

You can prove $\frac{d}{dx}[\cos x]=-\sin x$ using the same method and the same two limits above.

Example 4: Find the derivative of tan *x* using the quotient rule and the derivatives of sin *x* and cos *x*.

We now have the power rule, the product rule, the quotient rule, and the derivatives of all 6 trig functions at our disposal.

Example 5: Find the derivative of each function. Before you begin, state which rule(s) you are going to have to use. The product rule seems to be the rule that people forget to use … try not to be one of those people! ☺

a) $f(x) = x^2 \sin x$

b) $f(x) = \dfrac{\cos x}{x}$

c) $g(t) = \sqrt{t} + 4\sec t$

d) $h(\theta) = 5\sec\theta + \tan\theta$

e) $h(s) = \dfrac{1}{s} - 10\csc s$

f) $y = x\cot x$

Using the TI – 83+

You can use your calculator to graph the derivative for you using the procedures outlined below. This is not a required skill for success in this course, just something else your calculator can do. IF you have a TI-84, this process will be much quicker. I will leave this page for you to read if you are interested.

We will be using the `nDeriv(` function, except we will be using it to define a function under Y_1. Remember the syntax is

`nDeriv(function,variable,value)`

For our example, let's use cos *x*. The difference will be that our function will be entered as a function of any variable other than *x*, and differentiated with respect to that same variable, and evaluated at the value of *x* instead of an actual number.

So, we want to enter the following:

Y_1 = `nDeriv(cos(T),T,X)`

Step 1: Press [Y=].

Step 2: To enter `nDeriv(` … Press [MATH] then 8: `nDeriv(`

Step 3: Enter the function using the [ALPHA] key to enter a variable other than *x*. I chose *T*, but it should work with any letter you choose.

Step 4: After entering a comma, enter the same letter from step 3 as the variable you want to take the derivative with respect to. Again, I chose *T*, it's completely your choice.

Step 5: Enter the last comma, and then push the [X,T,Θ,*n*] button to enter the *X*. Do NOT use the [ALPHA] key to enter the *X* or it will not work.

Step 6: Graph the function by pressing [GRAPH].

Example 6: Graph the derivative of $f(x) = \ln x$. What function does this look like? Graph your guess on the same screen.

Example 7: Graph the derivative of $f(x) = e^x$. What function does this look like? Graph your guess on the same screen.