5.3 DEFINITE INTEGRALS AND ANTIDERIVATIVES

Notecards from Section 5.3: Properties of a Definite Integral; Average Value of $f(x)$; Relationship between Average Value of $f(x)$ and Average Rate of Change

In the last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If you reverse the *order* of integration you get the opposite answer.

2. Zero: $\int_a^a f(x)\,dx = 0$

This should make sense if you think about the "area" of a rectangle with no width.

3. Constant Multiple: If k is any constant, then $\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

This allows you to integrate functions that are added or subtracted separately. Notice, there are NO rules here for two functions that are multiplied or divided ... that comes later!

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Pay close attention to the limits of integration ... this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Example 1: Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find the following:

a) $\int_2^6 [f(x)+g(x)]\,dx$

b) $\int_2^6 [g(x)-f(x)]\,dx$

c) $\int_2^6 3f(x)\,dx$

d) $\int_2^6 (f(x)+2)\,dx$

Example 2: Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find the following:

a) $\int_0^7 f(x)\,dx$

b) $\int_5^0 f(x)\,dx$

c) $\int_5^5 f(x)\,dx$

d) $\int_0^5 3f(x)\,dx$

Average Value of a Function

Suppose you wanted to find the average temperature during a 24 hour period. How could you do it?

Suppose $f(t)$ represents the temperature at time t, measured in hours since midnight. One way to start is to measure the temperature at n equally spaced times $t_1, t_2, t_3, \ldots t_n$ and then average those temperatures.

Example 3: Using this method, write an expression for the AVERAGE temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be $\Delta t = \frac{24}{n}$, so $n = \frac{24}{\Delta t}$.

Example 4: Substitute this value of n into your expression above and simplify.

Example 5: The last expression gives us an approximate Average Temperature. As $n \to \infty$ (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the Definite Integral that gives us the Average Temperature since midnight.

Example 6: Do you think that there is any point during the day that the temperature reading on the thermometer is the *exact* value of the average temperature?

The process that we just used to find the average temperature is used to find the Average Value of any function.

The Average Value of a Function

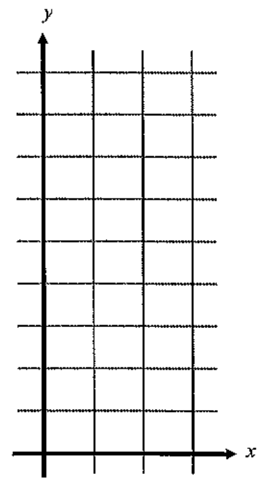
If f is integrable on $[a, b]$, its **average value** on $[a, b]$ is given by

$$\text{AVERAGE VALUE} = \frac{1}{b-a}\int_a^b f(x)\,dx \;\ldots \text{ or } \ldots\; \text{AVERAGE VALUE} = \frac{\int_a^b f(x)\,dx}{b-a}$$

The *average value of a function* is just … "the integral over the interval".

To get a more geometric idea of what the average value is, complete the following examples:

Example 7: Graph the function $y = x^2$ on [0,3] on the grid to the right.



Example 8: Set up a definite integral to find the average value of y on [0, 3], then use your calculator to evaluate the definite integral.

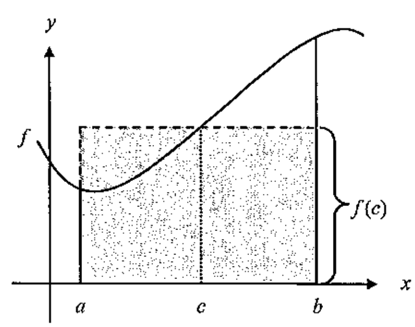
Example 9: Graph this as a value as a function on the grid to the right. Does the function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

Example 10: What do you suppose is the relationship between the area between the x – axis and the curve $y = x^2$ on [0, 3] and the area of the rectangle formed using the average value as the height and the interval [0, 3] as the width?

The Mean Value Theorem for Definite Integrals … see the connection … Mean … Average …

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between $x = a$ and $x = b$, then there is *some* number c between a and b whose function value you can use to form a rectangle that has an area equal to the area under the curve.

Example 11: What is an expression that could be used to determine the *area under the curve* from a to b ?



Example 12: What is the area of the shaded rectangle?

This value of $f(c)$ is just the *Average Value* of f on the interval $[a, b]$.

So another way to look at this is the Mean Value Theorem for Integrals just says that at some point within the interval the function MUST equal its average value.

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

...Once again, we have a theorem that tells us a value of c exists, but the theorem doesn't actually find it for us!

It is greatly important that you understand the difference between *average rate of change* and *average value.*

More on this after we finish §5.4. For now, understand that the average rate of change is simply the "slope between two points" on a given function and the average value of the function is the "integral divided by the interval".