31. $f(x) = \lim_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0^+} \dfrac{3(1+h)-2-2}{h}$

$= \lim_{h\to 0^+} \dfrac{3h-1}{h} = -\infty$

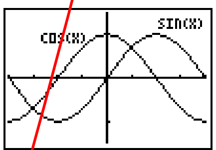
Does not exist.

32. We show that the right-hand derivative at 1 does not exist.

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{3(1+h)-(1)^3}{h}$$

$$= \lim_{h\to 0^+} \frac{2+3h}{h} = \lim_{h\to 0^+} \left(\frac{2}{h}+3\right) = \infty$$

33.



$[-\pi, \pi]$ by $[-1.5, 1.5]$

The cosine function could be the derivative of the sine function. The values of the cosine are positive where the sine is increasing, zero where the sine has horizontal tangents, and negative where sine is decreasing.

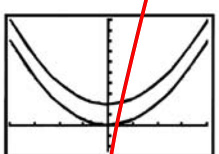
34. $\lim_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \dfrac{\sqrt{h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \dfrac{\sqrt{h}}{h}$

$= \lim_{h\to 0^+} \dfrac{1}{\sqrt{h}} = \infty$

Thus, the right-hand derivative at 0 does not exist.

35. Two parabolas are parallel if they have the same derivative at every value of x. This means that their tangent lines are parallel at each value of x.

Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.



[–4, 4] by [–5, 20]

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

36. True. $f'(x) = 2x + 1$

37. False. Let $f(x) = \dfrac{|x|}{x}$. The left hand derivative at $x = 0$ is -1 and the right hand derivative at $x = 0$ is 1. $f'(0)$ does not exist.

38. C. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{4-3(x+h)-(4+3x)}{h}$

$= \lim_{h\to 0} \dfrac{-3h}{h} = -3$

39. A. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{1-3(x+h)^2-(1-3x^2)}{h}$

$= \lim_{h\to 0} \dfrac{-6xh-3h^2}{h} = \lim_{h\to 0} 6x+h = -6x$

$-6(1) = -6.$

40. B. $\lim_{h\to 0^-} \dfrac{(0-1+h)^2-(0-1)}{h}$

$= \lim_{h\to 0^-} \dfrac{2-2h+h^2}{h} = \lim_{h\to 0} h = 0$

41. C. $\lim_{h\to 0^+} \dfrac{(2(0+h)-1)-(2(0)-1)}{h}$

$= \lim_{h\to 0^+} \dfrac{2h}{h} = 2$

42. **(a)** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \dfrac{2xh+h^2}{h}$

$= \lim_{h\to 0} (2x+h) = 2x$

(b) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{2(x+h)-2x}{h} = \lim_{h\to 0} 2 = 2$

(c) $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2x = 2(1) = 2$

(d) $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} 2 = 2$

(e) Yes, the one-sided limits exist and are the same, so $\lim_{x\to 1} f'(x) = 2.$

(f) $\lim_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \dfrac{(1+h)^2-1^2}{h}$

$= \lim_{h\to 0^-} \dfrac{2h+h^2}{h} = \lim_{h\to 0^-} (2+h) = 2$

(g) $\lim_{h\to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \dfrac{2(1+h)-1^2}{h}$

$= \lim_{h\to 0^-} \dfrac{1+2h}{h} = \lim_{h\to 0^-} \left(\dfrac{1}{h}+2\right) = -\infty$

The right-hand derivative does not exist.

39. Continued

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{3-(1+h)-2}{h}$$
$$= \lim_{h\to 0^-}(-1) = -1$$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{a(1+h)^2+b(1+h)-2}{h}$$
$$= \lim_{h\to 0^-}\frac{a+2ah+ah^2+b+bh-2}{h}$$
$$= \lim_{h\to 0^-}\frac{2ah+ah^2+bh+(a+b-2)}{h}$$
$$= \lim_{h\to 0^-}(2a+ah+b)$$
$$= 2a+b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$. Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

40. True. See Theorem 1.

41. False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

42. B.

43. A. $\displaystyle\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$

$$= \frac{\sqrt[3]{1.001-1} - \sqrt[3]{0.999-1}}{0.002}$$
$$= \infty$$
$$f(1) = \sqrt[3]{1-1} = 0$$

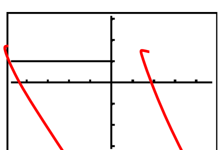
44. B. $\displaystyle\lim_{h\to 0^-}\frac{2(0+h)+1-(2(0)+1)}{h}$

$$= \lim_{h\to 0}\frac{2h}{h} = 2$$

45. C. $\displaystyle\lim_{h\to 0}\frac{(0+h)^2+1-(0^2+1)}{h}$

$$= \lim_{h\to 0}\frac{h^2}{h} = 0$$

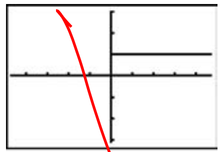
46. (a)



[−4.7, 4.7] by [−3.1, 3.1]

(b) You can use Trace to help see that the value of Y1 is 1 for every $x < 0$ and is 0 for every $x \ge 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0. \end{cases}$

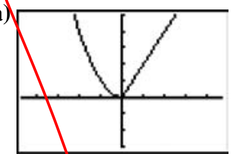
(c)



[−4.7, 4.7] by [−3.1, 3.1]

(d) You can use Trace to help see that the value of Y1 is 0 for every $x < 0$ and is 1 for every $x \ge 0$. It appears to be the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0. \end{cases}$

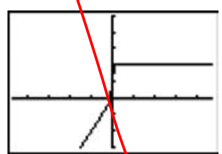
47. (a)



[−4.7, 4.7] by [−3, 5]

(b) See exersize 46.

(c)



[−4.7, 4.7] by [−3, 5]

(d) NDER (Y1, x, −0.1) = −0.1, NDER(Y1, x, 0) = 0.9995, NDER (Y1, x, 0.1) = 2.

48. (a) Note that $-x \le \sin\frac{1}{x} \le x$, for all x, so $\displaystyle\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$ by the Sandwich Theorem. Therefore, f is continuous at $x = 0$.

(b) $\displaystyle\frac{f(0+h)-f(0)}{h} = \frac{h\sin\frac{1}{h}-0}{h} = \sin\frac{1}{h}$

(c) The limit does not exist because $\sin\frac{1}{h}$ oscillates between −1 and 1 an infinite number of times arbitrarily close to $h = 0$ (that is, for h in any open interval containing 0).

(d) No, because the limit in part (c) does not exist.

(e) $\displaystyle\frac{g(0+h)-g(0)}{h} = \frac{h^2\sin\left(\frac{1}{h}\right)-0}{h} = h\sin\frac{1}{h}$

As noted in part (a), the limit of this as x approaches zero is 0, so $g'(0) = 0$.

Section 3.3 Rules for Differentiation (pp. 116–126)

Quick Review 3.3

1. $(x^2-2)(x^{-1}+1) = x^2x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
$= x + x^2 - 2x^{-1} - 2$

50. If the radius of a sphere is changed by a very small amount Δr, the change in the volume can be thought of as a very thin layer with an area given by the surface area, $4\pi r^2$, and a thickness given by Δr. Therefore, the change in the volume can be thought of as$(4\pi r^2)(\Delta r)$, which means that the change in the volume divided by the change in the radius is just $4\pi r^2$.

51. Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree x years from now. Then $t(0) = 156$, $y(0) = 12$, $t'(0) = 13$, and $y'(0) = 1.5$. The rate of increase of production is

$$\frac{d}{dx}(ty) = t(0)y'(0) + y(0)t'(0) = (156)(1.5) + (12)(13) = 390$$

bushels of annual production per year.

52. Let $m(x)$ be the number of members and $c(x)$ be the pavillion cost x years from now. Then $m(0) = 65$, $c(0) = 250$, $m'(0) = 6$, and $c'(0) = 10$. The rate of change of each member's share is $\frac{d}{dx}\left(\frac{c}{m}\right) = \frac{m(0)c'(0) - c(0)m'(0)}{[m(0)]^2}$

$$= \frac{(65)(10) - (250)(6)}{(65)^2} \approx -0.201 \text{ dollars per year. Each}$$

member's share of the cost is decreasing by approximately 20 cents per year.

3.3

53. False. π is a constant so $\frac{d}{dx}(\pi) = 0$.

54. True. $f'(x) = -\frac{1}{x^2}$ is never zero, so there are no horizontal tangents.

55. B. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= (2)(1) + (-1)(3)$$
$$= -1$$

56. D. $$f(x) = x - \frac{1}{x}$$
$$f'(x) = 1 + \frac{1}{x^2}$$
$$f''(x) = -\frac{2}{x^3}$$

57. E. $$\frac{d}{dx}\left(\frac{x+1}{x^2-1}\right)$$
$$= \frac{(x-1)-(x+1)}{(x-1)^2}$$
$$= \frac{-2}{(x-1)^2}$$

58. B. $f(x) = (x^2 - 1)(x^2 + 1)$
$x = \pm 1$

59. (a) It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

(b) It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

(c) $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$. This is equivalent to the product rule given in the text.

(d) Because dx is "infinitely small," and this could be thought of as dividing by zero.

(e) $$d\left(\frac{u}{v}\right) = \frac{u+du}{v+dv} - \frac{u}{v}$$
$$= \frac{(u+du)(v) - (u)(v+dv)}{(v+dv)(v)}$$
$$= \frac{uv + vdu - uv - udv}{v^2 + vdv}$$
$$= \frac{vdu - udv}{v^2}$$

Quick Quiz Sections 3.1–3.3

1. D.

2. A. Since $\frac{dy}{dx}$ gives the slope:

$$m_1 = \frac{1-2}{-1-1} = \frac{1}{2}$$
$$m_2 = -\frac{1}{m_1} = -2$$

Pg.

126

3. C. $$\frac{dy}{dx} = \frac{d}{dx}\frac{4x-3}{2x+1}$$
$$= \frac{4(2x+1) - 2(4x-3)}{(2x-1)^2}$$
$$= \frac{10}{(2x-1)^2}$$

4. (a) $$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 4x^2)$$
$$= 4x^2 - 8x = 0$$
$$x = 0,\ \pm\sqrt{2}$$

(b) $x = 1 \qquad y = (1)^4 - 4(1)^2$
$y = -3$
$y = m(x - x_1) + y_1$
$y = -4(x - 1) - 3$
$y = -4x + 1$

4. Continued

(c) $m_2 = -\frac{1}{m_1} = \frac{1}{4}$

$$y = \frac{1}{4}(x-1) - 3$$
$$= \frac{1}{4}x - \frac{1}{4} - 3 = \frac{1}{4}x - \frac{13}{4}$$

Section 3.4 Velocity and Other Rates of Change (pp. 127–140)

Exploration 1 Growth Rings on a Tree

1. Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.

2. Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.

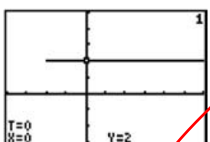
3. Since change in area is constant, so also is $\frac{\text{change in area}}{2\pi}$.

If we denote this latter constant by k, we have

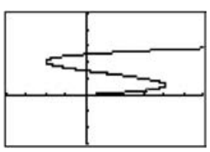
$\frac{k}{\text{change in radius}} = r$, which means that r varies inversely as the change in the radius. In other words, the change in radius must get smaller when r gets bigger, and vice-versa.

Exploration 2 Modeling Horizontal Motion

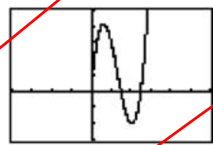
1. The particle reverses direction at about $t = 0.61$ and $t = 2.06$.



2. When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.

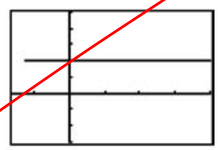


3. When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of t for when the particle reverses direction.

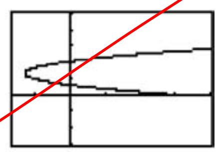


4. We can represent the velocity by graphing the parametric equations

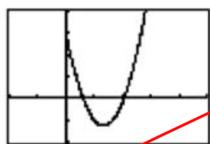
$x_4(t) = x_1'(t) = 12t^2 - 32t + 15,\ y_4(t) = 2$ (part 1),
$x_5(t) = x_1'(t) = 12t^2 - 32t + 15,\ y_5(t) = t$ (part 2),
$x_6(t) = t,\ y_6(t) = x_1'(t) = 12t^2 - 32t + 15$ (part 3)



$[-8, 20]$ by $[-3, 5]$
(x_4, y_4)



$[-8, 20]$ by $[-3, 5]$
(x_5, y_5)



$[-2, 5]$ by $[-10, 20]$
(x_6, y_6)

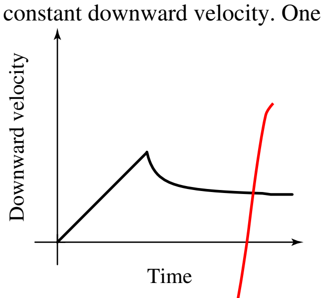
For (x_4, y_4) and (x_5, y_5), the particle is moving to the right when the x-coordinate of the graph (velocity) is positive, moving to the left when the x-coordinate of the graph (velocity) is negative, and is stopped when the x-coordinate of the graph (velocity) is 0. For (x_6, y_6),the particle is moving to the right when the y-coordinate of the graph (velocity) is positive, moving to the left when the y-coordinate of the graph (velocity) is negative, and is stopped when the y-coordinate of the graph (velocity) is 0.

Exploration 3 Seeing Motion on a Graphing Calculator

1. Let tMin = 0 and tMax = 10.

2. Since the rock achieves a maximum height of 400 feet, set yMax to be slightly greater than 400, for example yMax = 420.

33. Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:



3.4

34. (a) $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\frac{dv}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of volume per foot of radius.

(b) The increase in the volume is

$\frac{4}{3}\pi(2.2)^3 - \frac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

35. Let v_0 be the exit velocity of a particle of lava. Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is $\frac{ds}{dt} = v_0 - 32t$. Solving $\frac{ds}{dt} = 0$ gives $t = \frac{v_0}{32}$. Then the maximum height, in feet, is

$s\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^{\ 2}}{64}$. Solving

$\frac{v_0^{\ 2}}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was about 348.712 ft/sec. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \bullet \frac{1 \text{ mi}}{5280 \text{ ft}}$, we find that this is equivalent to about 237.758 mi/h.

36. By estimating the slope of the velocity graph at that point.

37. The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in $[-6, 8]$ by $[-3, 5]$.

(a) It begins at the point $(-5, 2)$ moving in the positive direction. After a little more than one second, it has moved a bit past $(6, 2)$ and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near $(-2, 2)$ and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

(b) The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \le t \le 2.167$ and $t \ge 3.180$. It slows down during the approximate intervals $0 \le t \le 1.153$ and $2.167 \le t \le 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed, $|\text{NDER } x(t)|$, using function mode in the window $[0, 5]$ by $[0, 10]$.

(c) The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(d) The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(e) The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase.
The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

(f) The particle is at (5, 2) when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

38. (a) Solving $160 = 490t^2$ gives $t = \pm\frac{4}{7}$. It took $\frac{4}{7}$ of a second.

The average velocity was $\frac{160 \text{ cm}}{\left(\frac{4}{7}\right)}$ sec = 280 cm/sec.

39. Since profit = revenue – cost, the Sum and Difference Rule gives $\frac{d}{dx}(\text{profit}) = \frac{d}{dx}(\text{revenue}) - \frac{d}{dx}(\text{cost})$, where x is the number of units produced. This means that marginal profit = marginal revenue – marginal cost.

40. False. It is the absolute value of the velocity.

41. True. The acceleration is the first derivative of the velocity which, in turn, is the second derivative of the position function.

42. C. $\frac{d}{dx}\frac{x^2-2}{x+4} = \frac{(x+4)(2x)-(x^2-2)}{x+4}$

$f'(x) = \frac{x^2+8x-2}{x+4}$

$f'(-1) = \frac{(-1)^2+8(-1)-2}{(-1)+4} = 0$

3.4

43. D. $V(x) = x^3$

$\frac{dv}{dx} = 3x^2$

44. E. $\frac{ds}{dt} = \frac{d}{dt}(2+7t-t^2)$

$v(t) = 7 - 2t < 0$

$7 < 2t$

$\frac{7}{2} < t$

$4 > \frac{7}{2}$

45. C. $v(t) = 7 - 2t = 0$

$7 = 2t$

$t = \frac{7}{2}$

38. Observe the pattern:

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d^5}{dx^5}\cos x = -\sin x$$
$$\frac{d^2}{dx^2}\cos x = -\cos x \qquad \frac{d^6}{dx^6}\cos x = -\cos x$$
$$\frac{d^3}{dx^3}\cos x = \sin x \qquad \frac{d^7}{dx^7}\cos x = \sin x$$
$$\frac{d^4}{dx^4}\cos x = \cos x \qquad \frac{d^8}{dx^8}\cos x = \cos x$$

Continuing the pattern, we see that

$\frac{d^n}{dx^n}\cos x = \sin x$ when $n = 4k + 3$ for any whole number k.

Since $999 = 4(249) + 3$, $\frac{d^{999}}{dx^{999}}\cos x = \sin x$.

39. Observe the pattern:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d^5}{dx^5}\sin x = \cos x$$
$$\frac{d^2}{dx^2}\sin x = -\sin x \qquad \frac{d^6}{dx^6}\sin x = -\sin x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x \qquad \frac{d^7}{dx^7}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\sin x = \sin x \qquad \frac{d^8}{dx^8}\sin x = \sin x$$

Continuing the pattern, we see that

$\frac{d^n}{dx^n}\sin x = \cos x$ when $n = 4k+1$ for any whole number k.

Since $725 = 4(181) + 1$, $\frac{d^{725}}{dx^{725}}\sin x = \cos x$.

40. The line is tangent to the graph of $y = \sin x$ at (0, 0). Since $y'(0) = \cos(0) = 1$, the line has slope 1 and its equation is $y = x$.

41. (a) Using $y = x$, $\sin(0.12) \approx 0.12$.

(b) $\sin(0.12) \approx 0.119712$2; The approximation is within 0.0003 of the actual value.

42.
$$\begin{aligned}\frac{d}{dx}\sin 2x &= \frac{d}{dx}(2\sin x\cos x)\\ &= 2\frac{d}{dx}(\sin x\cos x)\\ &= 2\left[(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\sin x)\right]\\ &= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]\\ &= 2(\cos^2 x - \sin^2 x)\\ &= 2\cos 2x\end{aligned}$$

43.
$$\begin{aligned}\frac{d}{dx}\cos 2x &= \frac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]\\ &= \left[(\cos x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\cos x)\right] -\\ &\quad \left[(\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)\right]\\ &= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)\\ &= -4\sin x\cos x\\ &= -2(2\sin x\cos x)\\ &= -2\sin 2x\end{aligned}$$

p147

44. True. The derivative is positive at $t = \frac{3\pi}{4}$.

45. False. The velocity is negative and the speed is positive at $t = \frac{\pi}{4}$.

46. A.
$$\begin{aligned} y &= \sin x + \cos x\\ \frac{dy}{dx_1} &= \frac{d}{dx} = \cos x - \sin x\\ y(\pi) &= \sin\pi + \cos\pi = -1\\ y'(\pi) &= \cos\pi - \sin\pi = -1\\ y &= -1(x-\pi) - 1\\ y &= -x + \pi - 1\end{aligned}$$

p147

47. B. See 46.
$$m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$
$$y = (x-\pi) - 1$$

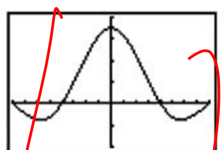
48. C.
$$\begin{aligned} y &= x\sin x\\ y' &= \sin x + x\cos x\\ y'' &= \cos x + \cos x - x\sin x\\ &= -x\sin x + 2\cos x\end{aligned}$$

49. C. $v(t) = \frac{ds}{dt} = \frac{d}{dt}(3 + \sin t)$

$v(t) = \cos t = 0$

$t = \frac{\pi}{2}$

50. (a)



[–360, 360] by [–0.01, 0.02]

The limit is $\frac{\pi}{180}$ because this is the conversion factor for changing from degrees to radians.

P155

63. Continued

Acceleration: $s''(t) = \dfrac{d}{dt}\dfrac{2}{\sqrt{1+4t}}$

$$= \frac{(\sqrt{1+4t})\dfrac{d}{dt}(2) - 2\dfrac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$$

$$= \frac{-2\left(\dfrac{1}{2\sqrt{1+4t}}\right)\dfrac{d}{dt}(1+4t)}{1+4t}$$

$$= \frac{\dfrac{-4}{\sqrt{1+4t}}}{1+4t} = -\frac{4}{(1+4t)^{3/2}}$$

At $t = 6$, the acceleration is $-\dfrac{4}{[1+4(6)]^{3/2}} = -\dfrac{4}{125}\ \text{m/sec}^2$

64. Acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \left(\dfrac{dv}{ds}\right)(v) = \left[\dfrac{d}{ds}(k\sqrt{s})\right](k\sqrt{s})$

$= \left(\dfrac{k}{2\sqrt{s}}\right)(k\sqrt{s}) = \dfrac{k^2}{2}$, a constant.

65. Note that this Exercise concerns itself with the slowing down caused by the earth's atmosphere, *not* the acceleration caused by gravity.

Given: $v = \dfrac{k}{\sqrt{s}}$

$$\text{Acceleration } = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = (v)\left(\frac{dv}{ds}\right)$$

$$= \left(\frac{k}{\sqrt{s}}\right)\frac{d}{ds}\frac{k}{\sqrt{s}}$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\sqrt{s}\dfrac{d}{ds}(k) - k\dfrac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right)$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\dfrac{-k}{(2\sqrt{s})}}{s}\right)$$

$$= -\frac{k^2}{2s^2},\ s \ge 0$$

Thus, the acceleration is inversely proportional to s^2.

66. Acceleration $= \dfrac{dv}{dt} = \dfrac{df(x)}{dt} = \dfrac{df(x)}{dx}\dfrac{dx}{dt} = f'(x)f(x)$

67. $\dfrac{dT}{du} = \dfrac{dT}{dL}\dfrac{dL}{du} = \left(\dfrac{d}{dL}2\pi\sqrt{\dfrac{L}{g}}\right)(kL)$

$$= \left(2\pi\frac{1}{2\sqrt{\dfrac{L}{g}}}\right)\left(\frac{d}{dL}\frac{L}{g}\right)(kL)$$

$$= \left(\frac{\pi}{\sqrt{\dfrac{L}{g}}}\right)\left(\frac{1}{g}\right)(kL) = k\pi\sqrt{\frac{L}{g}} = \frac{kT}{2}$$

68. No, this does not contradict the Chain Rule. The Chain Rule states that if two functions are differentiable at the appropriate points, then their composite must also be differentiable. It does not say: If a composite is differentiable, then the functions which make up the composite must all be differentiable.

69. Yes. Note that $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

70. False. See example 8.

71. False. It is +1.

72. E. $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan(4x)$

$y = \tan u \qquad u = 4x$

$\dfrac{dy}{du} = \sec^2 u \qquad \dfrac{du}{dx} = 4$

$\dfrac{dy}{dx} = 4\sec^2(4x)$

73. C. $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos^2(x^3+x^2)$

$y = \cos^2 u \qquad u = x^3 + x^2$

$\dfrac{dy}{du} = -2\sin u\cos u \qquad \dfrac{du}{dx} = 3x^2 + 2x$

$\dfrac{dy}{dx} = -2(3x^2+2x)\cos(3x^2+2x)\sin(3x^2+2x)$

74. A. $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dy}{dt}}$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(-1+\sin t)$

$= \cos t$

$\dfrac{dx}{dt} = \dfrac{d}{dt}(t - \cos t)$

$= 1 + \sin t$

$\dfrac{\dfrac{dy}{dt}}{\dfrac{dy}{dt}} = \dfrac{\cos t}{1+\sin t}$

$\left.\dfrac{dy}{dx}\right|_{t=0} = \dfrac{\cos 0}{1+\sin 0} = 1$

$x(0) = 0 - \cos 0 = -1$

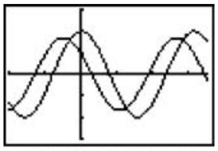
$y(0) = -1 + \sin 0 = -1$

$y = 1(x - (-1) + (-1)) = x$

75. B. See problem 74.

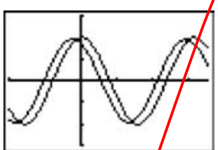
$$\frac{dy}{dx} = \frac{\cos t}{1+\sin t} = 0$$
$$\cos t = 0$$
$$t = \frac{\pi}{2}$$

76. For $h = 1$:



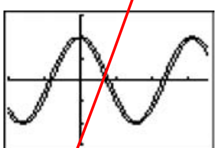
[−2, 3.5] by [−3, 3]

For $h = 0.5$:



[−2, 3.5] by [−3, 3]

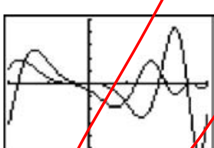
For $h = 0.2$:



[−2, 3.5] by [−3, 3]

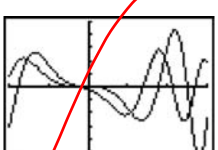
As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = 2\cos 2x$). This is because $2\cos 2x$ is the derivative of $\sin 2x$, and the second curve is the difference quotient used to define the derivative of $\sin 2x$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

77. For $h = 1$:



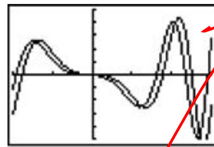
[−2, 3] by [−5, 5]

For $h = 0.3$:



[−2, 3] by [−5, 5]

For $h = 0.3$:



[−2, 3] by [−5, 5]

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x\sin(x^2)$). This is because $-2x\sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

78. (a) Let $f(x) = |x|$,

Then

$$\frac{d}{dx}|u| = \frac{d}{dx}f(u) = f'(u)\frac{du}{dx} = \left(\frac{d}{dx}|u|\right)\left(\frac{du}{dx}\right) = \frac{u}{|u|}u'.$$

The derivative of the absolute value function is +1 for positive values, −1 for negative values, and undefined at 0. So $f'(u) = \begin{cases} -1, & u < 0 \\ 1, & u > 0. \end{cases}$

But this is exactly how the expression $\frac{u}{|u|}$ evaluates.

(b) $$f'(x) = \left[\frac{d}{dx}(x^2-9)\right] \bullet \frac{x^2-9}{|x^2-9|} = \frac{(2x)(x^2-9)}{|x^2-9|}$$

$$g'(x) = \frac{d}{dx}(|x|\sin x)$$
$$= |x|\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}|x|$$
$$= |x|\cos x + \frac{x\sin x}{|x|}$$

Note: The expression for $g'(x)$ above is undefined at $x = 0$, but actually

$$g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{|h|\sin h}{h} = 0.$$

Therefore, we may express the derivative as

$$g'(x) = \begin{cases} |x|\cos x + \dfrac{x\sin x}{|x|}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

79. $$\frac{dG}{dx} = \frac{d}{dx}\sqrt{uv} = \frac{d}{dx}\sqrt{x(x+c)} = \frac{d}{dx}\sqrt{x^2+cx}$$
$$= \frac{1}{2\sqrt{x^2+cx}}\frac{d}{dx}(x^2+cx) = \frac{2x+c}{2\sqrt{x^2+cx}} = \frac{x+(x+c)}{2\sqrt{x(x+c)}}$$
$$= \frac{u+v}{2\sqrt{uv}} = \frac{A}{G}$$

Quick Quiz Sections 3.4-3.6

1. B. $y = \sin^4 u \qquad u = 3x$

$$\frac{dy}{du} = 4\sin^3 u \cos u \qquad \frac{du}{dx} = 3$$
$$\frac{dy}{dx} = 12\sin^3(3x)\cos(3x)$$

p156

2. A.
$$y = \cos x + \tan x$$
$$y' = -\sin x + \sec^2 x$$
$$y'' = -\cos x + 2\sec^2 x \tan x$$

3. C. $x = 3\sin t \qquad y = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{dy}{dt} = \frac{d}{dt}(2\cos t) = -2\sin t$$
$$\frac{dx}{dt} = \frac{d}{dt}(3\sin t) = 3\cos t$$
$$\frac{dy}{dx} = \frac{-2\sin t}{3\cos t} = -\frac{2}{3}\tan t.$$

4. (a) $s(0) = -(0)^2 + (0) + 2$

$s(0) = 2$

$(0, 2)$

(b) $-t^2 + t + 2 > 0$

$(t+1)(-t+2) > 0$

$-1 < t < \frac{1}{2}$ but $t < 0$ not real

$0 \le t < \frac{1}{2}$

(c) $t > \frac{1}{2}$

(d) $v(t) = \frac{ds}{dt} = \frac{d}{dt}(-t^2 + t + 2)$
$= -2t + 1$

(e) $a(t) = \frac{dv}{dt} = \frac{d}{dt}(-2t + 1)$
$= -2$

(f) $t = \frac{1}{2}$

Section 3.7 Implicit Differentiation (pp. 157–164)

Exploration 1 An Unexpected Derivative

1. $2x - 2y - 2xy' + 2yy' = 0$. Solving for y', we find that $\frac{dy}{dx} = 1$ (provided $y \neq x$).

2. With a constant derivative of 1, the graph would seem to be a line with slope 1.

3. Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.

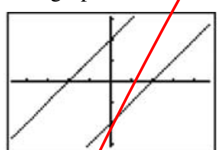
4. Factoring the original equation, we have

$[(x - y) - 2][(x - y) + 2] = 0$

$\therefore x - y - 2 = 0$ or $x - y + 2 = 0$

$\therefore y = x - 2$ or $y = x + 2$.

The graph is shown below.



[−4.7, 4.7] by [−3.1, 3.1]

5. At each point (x, y) on either line, $\frac{dy}{dx} = 1$. The condition $y \neq x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on x or y, but there are no inconsistencies.

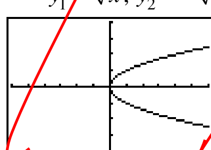
Quick Review 3.7

1. $x - y^2 = 0$

$x = y^2$

$\pm\sqrt{x} = y$

$y_1 = \sqrt{x},\ y_2 = -\sqrt{x}$



[-6, 6] by [-4, 4]

55. Continued

(c) The equation $x^3 + y^3 = 9xy$ is not affected by interchanging x and y, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the x-value and the y-value in the answer to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or approximately (4.762, 3.780).

56.

$$x^2 + 2xy - 3y^2 = 0$$
$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(0)$$
$$2x + 2x\frac{dy}{dx} + 2(y)(1) - 6y\frac{dy}{dx} = 0$$
$$(2x - 6y)\frac{dy}{dx} = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y} = \frac{x + y}{3y - x}$$

At (1, 1) the curve has slope $\frac{1+1}{3(1)-1} = \frac{2}{2} = 1$, so the normal line is $y = -1(x-1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for y in the original equation, we have:

$$x^2 + 2xy - 3y^2 = 0$$
$$x^2 + 2x(-x+2) - 3(-x+2)^2 = 0$$
$$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$$
$$-4x^2 + 16x - 12 = 0$$
$$-4(x-1)(x-3) = 0$$
$$x = 1 \text{ or } x = 3$$

Since the given point (1,1) had $x = 1$, we choose $x = 3$ and so $y = -(3) + 2 = -1$. The desired point is $(3, -1)$.

57.

$$xy + 2x - y = 0$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$
$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$
$$(x-1)\frac{dy}{dx} = -2 - y$$
$$\frac{dy}{dx} = \frac{-2-y}{x-1} = \frac{2+y}{1-x}$$

Since the slope of the line $2x + y = 0$ is -2, we wish to find points where the normal has slope -2, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have

$$\frac{2+y}{1-x} = \frac{1}{2}$$
$$2(2+y) = 1 - x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y-3)y + 2(-2y-3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y+1)(y+3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1, x = -2y - 3 = 2 - 3 = -1$.
At $y = -3: x = -2y - 3 = 6 - 3 = 3$.

The desired points are $(-1, -1)$ and $(3, -3)$.
Finally, we find the desired normals to the curve, which are the lines of slope -2 passing through each of these points.
At $(-1, -1)$, the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At $(3, -3)$, the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

58.

$$x = y^2$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$
$$1 = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at (x, y) has slope $-2y$. Thus, the normal line at (b^2, b) is $y = -2b(x - b^2) + b$, or $y = -2bx + 2b^3 + b$.

This line intersects the x-axis at $x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$, which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes ± 1, which gives

$-2y = \pm 1$ or $y = \pm\frac{1}{2}\left(\text{or } b = \pm\frac{1}{2}\right)$. The corresponding value of a is $b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal normals are perpendicular when $a = \frac{3}{4}$.

P164

59. False.

$$\frac{d}{dx}(xy^2 + x) = \frac{d}{dx}(1)$$
$$y^2 + 1 + 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-1-y}{2xy}, \quad \frac{-1-1}{2\left(\frac{1}{2}\right)1} = -2$$

60. True. By the power rule.

$$y = (x)^{1/3}$$
$$\frac{dy}{dx} = \frac{d}{dx}(x)^{1/3} = \frac{1}{3}x^{2/3} = \frac{1}{3x^{2/3}}$$

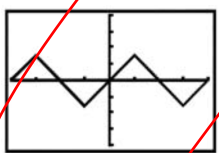
29. (a) Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus f is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so f has a differentiable inverse by Theorem 3.

(b) $f(0) = \cos 0 + 3(0) = 1;$
$f'(0) = -\sin 0 + 3 = 3$

(c) Since the graph of $y = f(x)$ includes the point (0, 1) and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include (1, 0) and the slope will be $\frac{1}{3}$,

Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \frac{1}{3}$.

30.



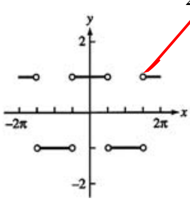
$[-2\pi, 2\pi]$ by $[-4, 4]$

(a) All reals

(b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) At the points $x = k\frac{\pi}{2}$, where k is an odd integer.

(d)



(e) $f'(x) = \frac{d}{dx}\sin^{-1}(\sin x)$

$$= \frac{1}{\sqrt{1-\sin^2 x}}\frac{d}{dx}\sin x$$
$$= \frac{\cos x}{\sqrt{1-\sin^2 x}}$$

which is ± 1 depending on whether $\cos x$ is positive or negative.

31. (a) $v(t) = \frac{dx}{dt} = \frac{1}{1+t^2}$ which is always positive.

(b) $a(t) = \frac{dv}{dt} = -\frac{2t}{(1+t^2)^2}$ which is always negative.

(c) $\frac{\pi}{2}$

32. $\frac{d}{dx}\cos^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right)$

$$= 0 - \frac{d}{dx}\sin^{-1}(x)$$
$$= -\frac{1}{\sqrt{1-x^2}}$$

33. $\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$

$$= 0 - \frac{d}{dx}\tan^{-1}(x)$$
$$= -\frac{1}{1+x^2}$$

34. $\frac{d}{dx}\csc^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1}(x)\right)$

$$= 0 - \frac{d}{dx}\sec^{-1}(x)$$
$$= -\frac{1}{|x|\sqrt{x^2-1}}$$

35. True. By definition of the Function.

36. False. The domain is all real numbers.

P171

37. E. $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)$

$y = \sin^{-1} x \qquad u = \frac{x}{2}$

$\sin y = x \qquad \frac{du}{dx} = \frac{1}{2}$

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$
$$\cos y\frac{dy}{dx} = 1$$
$$\frac{d}{dx} = \frac{1}{\cos y}$$
$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-\frac{x}{2}}}\frac{1}{2}$$
$$= \frac{1}{\sqrt{4-x^2}}$$

38. D. $\frac{d}{dx}\tan^{-1}(3x)$

$y = \tan^{-1} u \qquad u = 3x$

$\frac{d}{du}\tan y = \frac{d}{du}u \qquad \frac{du}{dx} = 3$

$$\sec^2 y\frac{dy}{du} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$= \frac{1}{1+(\tan y)^2}$$
$$= \frac{1}{1+u^2}\frac{du}{dx}$$
$$= \frac{3}{1+9x^2}$$

p171

39. A. $\frac{d}{dx}\sec^{-1}(x^2)$

$$y = \sec^{-1} u \qquad u = x^2$$
$$\frac{d}{du}(\sec y) = \frac{d}{du}u \qquad du = 2x$$
$$\sec y \tan y \frac{dy}{du} = 1$$
$$\frac{dy}{du} = \frac{1}{\sec y \tan y}$$
$$\frac{dy}{du} = \frac{1}{4\sqrt{u^2-1}}\frac{du}{dx}$$
$$= \frac{2x}{x^2\sqrt{x^4-1}}$$
$$= \frac{2}{x\sqrt{x^4-1}}$$

40. C. $\frac{d}{dx}\tan^{-1}(2x)$

See problem 38.

$$u = 2x$$
$$\frac{du}{dx} = 2$$
$$\frac{1}{1+u^2}\frac{du}{dx} = \frac{2}{1+4x^2}$$
$$\frac{2}{1+4(1)^2} = \frac{2}{5}$$

41. **(a)** $y = \frac{\pi}{2}$

(b) $y = -\frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \neq 0$.

42. **(a)** $y = 0$

(b) $y = \pi$

(c) None, since $\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2} \neq 0$.

43. **(a)** $y = \frac{\pi}{2}$

(b) $y = \frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

44. **(a)** $y = 0$

(b) $y = 0$

(c) None, since $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

45. **(a)** None, since $\sin^{-1} x$ is undefined for $x > 1$.

(b) None, since $\sin^{-1} x$ is undefined for $x < -1$.

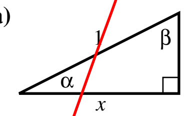
(c) None, since $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \neq 0$.

46. **(a)** None, since $\cos^{-1} x$ is undefined for $x > 1$.

(b) None, since $\cos^{-1} x$ is undefined for $x < -1$.

(c) None, since $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \neq 0$.

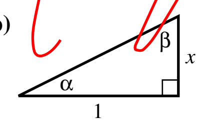
47. **(a)**



$\alpha = \cos^{-1} x$, $\beta = \sin^{-1} x$

So $\cos^{-1} x + \sin^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

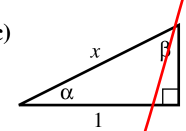
(b)



$\alpha = \tan^{-1} x$, $\beta = \cot^{-1} x$

So $\tan^{-1} x + \cot^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

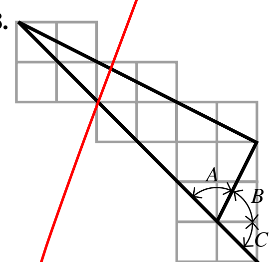
(c)



$\alpha = \sec^{-1} x$, $\beta = \csc^{-1} x$

So $\sec^{-1} x + \csc^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

48.



The “straight angle” with the arrows in it is the sum of the three angles A, B, and C.

A is equal to $\tan^{-1} 3$ since the opposite side is 3 times as long as the adjacent side.

B is equal to $\tan^{-1} 2$ since the side opposite it is 2 units and the adjacent side is one unit.

C is equal to $\tan^{-1} 1$ since both the opposite and adjacent sides are one unit long.

But the sum of these three angles is the “straight angle,” which has measure π radians.

P179

52. (a) $P(0)=\dfrac{200}{1+e^{5-0}}=1$

(b) $\dfrac{d}{dt}200((1+e^{5-t})^{-1})$

$=200(-1)(1+e^{5-t})^{-2}\dfrac{d}{dt}(1+e^{5-t})$

$=200(-1)(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$=\dfrac{200e^{5-t}}{(1+e^{5-t})^2}$

$P'(4)=\dfrac{200e^{5-4}}{(1+e^{5-4})^2}=39$

(c) $P'(5)=\dfrac{200e^{5-5}}{(1+e^{5-5})^2}=50$

53. $\dfrac{dA}{dt}=20\dfrac{d}{dt}\left(\dfrac{1}{2}\right)^{t/140}$

$=20\dfrac{d}{dt}2^{-t/140}$

$=20(2^{-t/140})(\ln 2)\dfrac{d}{dt}\left(-\dfrac{t}{140}\right)$

$=20(2^{-t/140})(\ln 2)\left(-\dfrac{t}{140}\right)$

$=-\dfrac{(2^{-t/140})(\ln 2)}{7}$

At $t=2$ days, we have $\dfrac{dA}{dt}=-\dfrac{(2^{-1/70})(\ln 2)}{7}\approx -0.098$ grams/day.
This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

54. (a) $\dfrac{d}{dx}\ln(kx)=\dfrac{1}{kx}\dfrac{d}{dx}kx=\dfrac{k}{kx}=\dfrac{1}{x}$

(b) $\dfrac{d}{dx}\ln(kx)=\dfrac{d}{dx}(\ln k+\ln x)$

$=0+\dfrac{d}{dx}\ln x=\dfrac{1}{x}$

55. (a) Since $f'(x)=2^x\ln 2$, $f'(0)=2^0\ln 2=\ln 2$.

(b) $f'(0)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{2^h-2^0}{h}$

$=\lim\limits_{h\to 0}\dfrac{2^h-1}{h}$

(c) Since quantities in parts (a) and (b) are equal,

$\lim\limits_{h\to 0}\dfrac{2^h-1}{h}=\ln 2.$

(d) By following the same procedure as above using $g(x)=7^x$, we may see that $\lim\limits_{h\to 0}\dfrac{7^h-1}{h}=\ln 7.$

56. Recall that a point (a, b) is on the graph of $y=e^x$ if and only if the point (b, a) is on the graph of $y=\ln x$. Since there are points (x, e^x) on the graph of $y=e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y=\ln x$ with arbitrarily large y-coordinates.

57. False. It is $\ln(2)2^x$.

58. False. It is $2e^{2x}$.

59. B. $P(0)=\dfrac{150}{1+e^{4-t}}=3$

60. D. $x+3>0$

$x>-3$

61. A. $y=\log_{10}(2x-3)$

$\dfrac{d}{dx}\log_a u=\dfrac{1}{u\ln a}\dfrac{du}{dx}$

$a=10$

$u=2x-3$

$\dfrac{du}{dx}=2$

$y'=\dfrac{2}{(2x-3)\ln 10}$

62. E. $y=2^{1-x}$

$\dfrac{d}{dx}a^u=a^u\ln a\dfrac{du}{dx}$

$a=2$

$u=1-x$

$\dfrac{du}{dx}=-1$

$y'=2^{1-x}\ln(2)(-1)$

$y'(2)=-2^{1-2}\ln(2)$

$y'(2)=-\dfrac{\ln(2)}{2}$

63. (a) The graph y_4 is a horizontal line at $y=a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y=\ln a$.

(c) $\dfrac{d}{dx}a^x=a^x$ if and only if $y_3=\dfrac{y_2}{y_1}=1$.

So if $y_3=\ln a$, then $\dfrac{d}{dx}a^x$ with equal a^x if and only if $\ln a=1$, or $a=e$.

(d) $y_2=\dfrac{d}{dx}a^x=a^x\ln a$. This will equal $y_1=a^x$ if and only if $\ln a=1$, or $a=e$.

64. $\dfrac{d}{dx}\left(-\dfrac{1}{2}x^2+k\right)=-x$ and $\dfrac{d}{dx}(\ln x+c)=\dfrac{1}{x}$.
Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

65. (a) Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Quick Quiz Sections 3.7–3.9

1. E. $y = \frac{9}{2x} - \frac{x^2}{2}$

$dy = -\frac{9}{2x^2} - x$

$dy = -\frac{9}{2(1)^2} - 1 = -\frac{11}{2}$

2. A. $dy = \frac{d}{dx}(\cos^3(3x-2))$

$dy = -9\cos^2(3x-2)\sin(3x-2)$

3. C. $dy = \frac{d}{dx}(\sin^{-1}(2x))$

$dy = \frac{2}{\sqrt{1-4x^2}}$

4. (a) Differentiate implicitly:

$$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$$

$$1 \cdot y^2 + x \cdot 2y\frac{dy}{dx} - \left(3x^2y + x^3\frac{dy}{dx}\right) = 0$$

$$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$$

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$$

(b) If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

at $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)^2(-2) - (-2)^2}{2(1)(-2) - (1)^3} = 2.$

The tangent line is $y + 2 = 2(x - 1)$.

At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)^2(3) - 3^2}{2(1)(3) - 1^3} = 0.$

The tangent line is $y = 3$.

(c) The tangent line is vertical where $2xy - x^3 = 0$, which implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve where $x = 0$. If $y = \frac{x^2}{2}$, then $x\left(\frac{x^2}{2}\right) - x^3\left(\frac{x^2}{2}\right) = 6.$

Then the only solution to this equation is $x = \sqrt[5]{-24}$.

Chapter 3 Review Exercises (pp. 181–184)

Red Ex.

1. $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2\sin x \cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2\sin^2 x + 2\cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2\sin x \cos x) = \frac{d}{dx}\sin 2x = (\cos 2x)(2)$

$= 2\cos 2x$

4. $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+1}{2x-1} = \frac{(2x-1)(2)-(2x-1)(2)}{(2x-1)^2} = -\frac{4}{(2x-1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt}\cos(1-2t) = -\sin(1-2t)(-2) = 2\sin(1-2t)$

6. $\frac{ds}{dt} = \frac{d}{dt}\cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2}\csc^2\left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x+1}) = (x)\left(\frac{1}{2\sqrt{2x+1}}\right)(2) + (\sqrt{2x+1})(1)$

$= \frac{x + (2x+1)}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta}\sec(1+3\theta) = \sec(1+3\theta)\tan(1+3\theta)(3)$

$= 3\sec(1+3\theta)\tan(1+3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta}\tan^2(3-\theta^2)$

$= 2\tan(3-\theta^2)\frac{d}{d\theta}\tan(3-\theta^2)$

$= 2\tan(3-\theta^2)\sec^2(3-\theta^2)(-2\theta)$

$= -4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2\csc 5x)$

$= (x^2)(-\csc 5x\cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2\csc 5x\cot 5x + 2x\csc 5x$

12. $\frac{dy}{dx}=\frac{d}{dx}\ln\sqrt{x}=\frac{1}{\sqrt{x}}\frac{d}{dx}\sqrt{x}=\frac{1}{\sqrt{x}}\bullet\frac{1}{2\sqrt{x}}=\frac{1}{2x},\ x>0$

13. $\frac{dy}{dx}=\frac{d}{dx}\ln(1+e^x)=\frac{1}{1+e^x}\frac{d}{dx}(1+e^x)=\frac{e^x}{1+e^x}$

14. $\frac{dy}{dx}=\frac{d}{dx}(xe^{-x})=(x)(e^{-x})(-1)+(e^{-x})(1)=-xe^{-x}+e^{-x}$

15. $\frac{dy}{dx}=\frac{d}{dx}(e^{1+\ln x})=\frac{d}{dx}(e^1e^{\ln x})=\frac{d}{dx}(ex)=e$

16. $\frac{dy}{dx}=\frac{d}{dx}\ln(\sin x)=\frac{1}{\sin x}\frac{d}{dx}(\sin x)=\frac{\cos x}{\sin x}=\cot x$, for values of x in the intervals $(k\pi,\ (k+1)\pi)$, where k is even.

17. $\frac{dr}{dx}=\frac{d}{dx}\ln(\cos^{-1}x)=\frac{1}{\cos^{-1}x}\frac{d}{dx}\cos^{-1}x$

$=\frac{1}{\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right)=-\frac{1}{\cos^{-1}x\sqrt{1-x^2}}$

18. $\frac{dr}{d\theta}=\frac{d}{d\theta}\log_2(\theta^2)=\frac{1}{\theta^2\ln 2}\frac{d}{d\theta}(\theta^2)=\frac{2\theta}{\theta^2\ln 2}=\frac{2}{\theta\ln 2}$

19. $\frac{ds}{dt}=\frac{d}{dt}\log_5(t-7)=\frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7)=\frac{1}{(t-7)\ln 5}$, $t>7$

20. $\frac{ds}{dt}=\frac{d}{dt}(8^{-t})=8^{-t}(\ln 8)\frac{d}{dt}(-t)=-8^{-t}\ln 8$

21. Use logarithmic differentiation.

$$y=x^{\ln x}$$
$$\ln y=\ln(x^{\ln x})$$
$$\ln y=(\ln x)(\ln x)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}(\ln x)^2$$
$$\frac{1}{y}\frac{dy}{dx}=2\ln x\frac{d}{dx}\ln x$$
$$\frac{dy}{dx}=\frac{2y\ln x}{x}$$
$$\frac{dy}{dx}=\frac{2x^{\ln x}\ln x}{x}$$

22. $\frac{dy}{dx}=\frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}}$

$$=\frac{\sqrt{x^2+1}\frac{d}{dx}\left[(2x)2^x\right]-(2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1}$$

$$=\frac{\sqrt{x^2+1}\left[(2x)(2^x)(\ln 2)+(2^x)(2)\right]-(2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$=\frac{(x^2+1)(2^x)(2x\ln 2+2)-2x^2(2^x)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\bullet 2^x)[(x^2+1)(x\ln 2+1)-x^2]}{(x^2+1)^{3/2}}$$

$$=\frac{(2\bullet 2^x)(x^3\ln 2+x^2+x\ln 2+1-x^2)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\bullet 2^x)(x^3\ln 2+x\ln 2+1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y=\frac{(2x)2^x}{\sqrt{x^2+1}}$$
$$\ln y=(2x)+\ln(2^x)-\ln\sqrt{x^2+1}$$
$$\ln y=\ln 2+\ln x+x\ln 2-\frac{1}{2}\ln(x^2+1)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}[\ln 2+\ln x+\ln 2-\frac{1}{2}\ln(x^2+1)]$$
$$\frac{1}{y}\frac{dy}{dx}=0+\frac{1}{x}+\ln 2-\frac{1}{2}\frac{1}{x^2+1}(2x)$$
$$\frac{dy}{dx}=y\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$
$$\frac{dy}{dx}=\frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$

23. $\frac{dy}{dx}=\frac{d}{dx}e^{\tan^{-1}x}=e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x=\frac{e^{\tan^{-1}x}}{1+x^2}$

24. $\frac{dy}{du}=\frac{d}{dx}\sin^{-1}\sqrt{1-u^2}$

$$=\frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\frac{d}{du}\sqrt{1-u^2}$$

$$=\frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u)=\frac{u}{|u|\sqrt{1-u^2}}$$

25. $\frac{dy}{dt}=\frac{d}{dt}\left(t\sec^{-1}t-\frac{1}{2}\ln t\right)$

$$=(t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right)+(\sec^{-1}t)(1)-\frac{1}{2t}$$

$$=\frac{t}{|t|\sqrt{t^2-1}}+\sec^{-1}t-\frac{1}{2t}$$

26. $\frac{dy}{dt}=\frac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$$=(1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2)+(\cot^{-1}2t)(2t)$$

$$=-\frac{2+2t^2}{1+4t^2}+2t\cot^{-1}2t$$

27. $\frac{dy}{dz} = \frac{d}{dz}(z\cos^{-1} z - \sqrt{1-z^2})$

$$= (z)\left(-\frac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1} z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z)$$

$$= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \frac{z}{\sqrt{1-z^2}}$$

$$= \cos^{-1} z$$

28. $\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$$= (2\sqrt{x-1})\left(-\frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2 - 1}}\right)\left(\frac{1}{2\sqrt{x}}\right) + (2\csc^{-1}\sqrt{x})\left(\frac{1}{2\sqrt{x-1}}\right)$$

$$= -\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

$$= -\frac{1}{x} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

29. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(\sec x)$

$$= \left(-\frac{1}{|\sec x|\sqrt{\sec^2 x - 1}}\right)\frac{d}{dx}(\sec x)$$

$$= -\frac{1}{|\sec x|\sqrt{\tan^2 x - 1}}\sec x \tan x$$

$$= -\frac{\sec x \tan x}{|\sec x \tan x|}$$

$$= -\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|} = -\frac{\sin x}{|\sin x|}$$

$$= \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Alternate method:

On the domain $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}$, $x \ne \frac{3\pi}{2}$, we may rewrite the function as follows:

$$y = \csc^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \sec^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \cos^{-1}(\cos x)$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi - x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi + x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Therefore, $\frac{dy}{dx} = \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

30. $\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1+\sin\theta}{1-\cos\theta}\right)^2$

$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{(1-\cos\theta)(\cos\theta) - (1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}\right)$$

$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1-\cos\theta)^2}\right)$$

$$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta - \sin\theta - 1}{(1-\cos\theta)^2}\right)$$

31. Since $y = \ln x^2$ is defined for all $x \ne 0$ and $\frac{dy}{dx} = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \ne 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and $\frac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$, the function is differentiable for all real x.

33. Since $y = \sqrt{\frac{1-x}{1+x^2}}$ is defined for all $x < 1$ and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1-x}{1+x^2}}}\frac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$$

$$= \frac{x^2 - 2x - 1}{2\sqrt{1-x}(1+x^2)^{3/2}},$$ which is defined only for $x < 1$, the function is differentiable for all $x < 1$.

34. Since $y = (2x-7)^{-1}(x+5) = \frac{x+5}{2x-7}$ is defined for all $x \ne \frac{7}{2}$ and $\frac{dy}{dx} = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2} = -\frac{17}{(2x-7)^2}$, the function is differentiable for all $x \ne \frac{7}{2}$.

35. Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$$\frac{dy}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$

$$x\frac{d}{dx} + (y)(1) + 2 + 3\frac{dy}{dx} = 0$$

$$(x+3)\frac{dy}{dx} = -(y+2)$$

$$\frac{dy}{dx} = -\frac{y+2}{x+3}$$

36. Use implicit differentiation.

$$5x^{4/5}+10y^{6/5}=15$$
$$\frac{d}{dx}(5x^{4/5})+\frac{d}{dx}(10y^{6/5})=\frac{d}{dx}(15)$$
$$4x^{-1/5}+12y^{1/5}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{4x^{-1/5}}{12y^{1/5}}=-\frac{1}{3(xy)^{1/5}}$$

37. Using implicit differentiation.

$$\sqrt{xy}=1$$
$$\frac{d}{dx}\sqrt{xy}=\frac{d}{dx}(1)$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx}+(y)(1)\right]=0$$
$$x\frac{dy}{dx}+y=0$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

Alternate method:

38. Use implicit differentiation.

$$y^2=\frac{x}{x+1}$$
$$\frac{d}{dx}y^2=\frac{d}{dx}\frac{x}{x+1}$$
$$2y\frac{dy}{dx}=\frac{(x+1)(1)-(x)(1)}{(x+1)^2}$$
$$\frac{dy}{dx}=\frac{1}{2y(x+1)^2}$$

39. $x^3+y^3=1$

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=\frac{d}{dx}(1)$$
$$3x^2+3y^2y'=0$$
$$y'=-\frac{x^2}{y^2}$$
$$y''=\frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)(y')}{y^4}$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4}$$
$$=-\frac{2xy^3+2x^4}{y^5}$$
$$=-\frac{2x(x^3+y^3)}{y^5}$$
$$=-\frac{2x}{y^5}$$

since $x^3+y^3=1$

40. $y^2=1-\frac{2}{x}$

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(1)-\frac{d}{dx}\left(\frac{2}{x}\right)$$
$$2yy'=\frac{2}{x^2}$$
$$y'=\frac{2}{x^{2(2y)}}=\frac{1}{x^2y}$$
$$y''=\frac{d}{dx}\left(\frac{1}{x^2y}\right)$$
$$=-\frac{1}{(x^2y)^2}\frac{d}{dx}(x^2y)$$
$$=-\frac{1}{(x^2y)^2}[(x^2)(y')+(y)(2x)]$$
$$=-\frac{1}{(x^2y)^2}\left[(x^2)\left(\frac{1}{x^2y}\right)+2xy\right]$$
$$=-\frac{1}{x^4y^2}\left(\frac{1}{y}+2xy\right)$$
$$=-\frac{1+2xy^2}{x^4y^3}$$

41. $y^3+y=2\cos x$

$$\frac{d}{dx}(y^3)+\frac{d}{dx}(y)=\frac{d}{dx}(2\cos x)$$
$$3y^2y'+y'=-2\sin x$$
$$(3y^2+1)y'=-2\sin x$$
$$y'=-\frac{2\sin x}{3y^2+1}$$
$$y''=\frac{d}{dx}\left(-\frac{2\sin x}{3y^2+1}\right)$$
$$=-\frac{(3y^2+1)(2\cos x)-(2\sin x)(6yy')}{(3y^2+1)^2}$$
$$=-\frac{(3y^2+1)(2\cos x)-(12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2+1)^2}$$
$$=-2\frac{(3y^2+1)^2\cos x+12y\sin^2 x}{(3y^2+1)^3}$$

42. $$x^{1/3}+y^{1/3}=4$$

$$\frac{d}{dx}(x^{1/3})+\frac{d}{dx}(y^{1/3})=\frac{d}{dx}(4)$$

$$\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0$$

$$y'=-\frac{x^{-2/3}}{y^{-2/3}}=-\left(\frac{y}{x}\right)^{2/3}$$

$$y''=\frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$

$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy'-(y)(1)}{x^2}\right)$$

$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right]-y}{x^2}\right)$$

$$=-\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3}-x^{-2}y)$$

$$=\frac{2}{3}x^{-4/3}y^{1/3}+\frac{2}{3}x^{-5/3}y^{2/3}$$

43. $y'=2x^3-3x-1,$

$y''=6x^2-3,$

$y'''=12x,$

$y^{(4)}=12,$ and the rest are all zero.

44. $y'=\dfrac{x^4}{24},$

$y''=\dfrac{x^3}{6},$

$6y'''=\dfrac{x^2}{2},$

$y^{(4)}=x,$

$y^{(5)}=1,$ and the rest are all zero.

45. $\dfrac{dy}{dx}=\dfrac{d}{dx}\sqrt{x^2-2x}=\dfrac{1}{2\sqrt{x^2-2x}}(2x-2)=\dfrac{x-1}{\sqrt{x^2-2x}}$

At $x=3$, we have $y=\sqrt{3^2-2(3)}=\sqrt{3}$

and $\dfrac{dy}{dx}=\dfrac{3-1}{\sqrt{3^2-2(3)}}=\dfrac{2}{\sqrt{3}}.$

(a) Tangent: $y=\dfrac{2}{\sqrt{3}}(x-3)+\sqrt{3}$ or $y=\dfrac{2}{\sqrt{3}}x-\sqrt{3}$

(b) Normal: $y=-\dfrac{\sqrt{3}}{2}(x-3)+\sqrt{3}$ or $y=-\dfrac{\sqrt{3}}{2}x+\dfrac{5\sqrt{3}}{2}$

46. $\dfrac{dy}{dx}=\dfrac{d}{dx}(4+\cot x-2\csc x)$

$=-\csc^2 x+2\csc x\cot x$

At $x=\dfrac{\pi}{2}$, we have

$y=4+\cot\dfrac{\pi}{2}-2\csc\dfrac{\pi}{2}=4+0-2=2$ and

$\dfrac{dy}{dx}=-\csc^2\dfrac{\pi}{2}+2\csc\dfrac{\pi}{2}\cot\dfrac{\pi}{2}=-1+2(1)(0)=-1.$

(a) Tangent: $y=-1\left(x-\dfrac{\pi}{2}\right)+2$ or $y=-x+\dfrac{\pi}{2}+2$

(b) Normal: $y=-1\left(x-\dfrac{\pi}{2}\right)+2$ or $y=x-\dfrac{\pi}{2}+2$

47. Use implicit differentiation.

$$x^2+2y^2=9$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(2y^2)=\frac{d}{dx}(9)$$

$$2x+4y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{4y}=-\frac{x}{2y}$$

Slope at (1, 2): $-\dfrac{1}{2(2)}=-\dfrac{1}{4}$

(a) Tangent: $y=-\dfrac{1}{4}(x-1)+2$ or $y=-\dfrac{1}{4}x+\dfrac{9}{4}$

(b) Normal: $y=4(x-1)+2$ or $y=4x-2$

48. Use implicit differentiation.

$$x+\sqrt{xy}=6$$

$$\frac{d}{dx}(x)+\frac{d}{dx}(\sqrt{xy})=\frac{d}{dx}(6)$$

$$1+\frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right)+(y)(1)\right]=0$$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx}=-1-\frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx}=\frac{2\sqrt{xy}}{x}\left(-1-\frac{y}{2\sqrt{xy}}\right)$$

$$=-2\sqrt{\frac{y}{x}}-\frac{y}{x}$$

Slope at (4, 1): $-2\sqrt{\dfrac{1}{4}}-\dfrac{1}{4}=-\dfrac{2}{2}-\dfrac{1}{4}=-\dfrac{5}{4}$

(a) Tangent: $y=-\dfrac{5}{4}(x-4)+1$ or $y=-\dfrac{5}{4}x+6$

(b) Normal: $y=-\dfrac{4}{5}(x-4)+1$ or $y=-\dfrac{4}{5}x-\dfrac{11}{5}$

49. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{-2\sin t}{2\cos t} = -\tan t$

At $t = \dfrac{3\pi}{4}$, we have $x = 2\sin\dfrac{3\pi}{4} = \sqrt{2}$,

$y = 2\cos\dfrac{3\pi}{4} = -\sqrt{2}$, and $\dfrac{dy}{dx} = -\tan\dfrac{3\pi}{4} = 1$.

The equation of the tangent line is

$y = 1(x - \sqrt{2}) + (-\sqrt{2})$, or $y = x - 2\sqrt{2}$.

50. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4\cos t}{-3\sin t} = -\dfrac{4}{3}\cot t$

At $t = \dfrac{3\pi}{4}$, we have $x = 3\cos\dfrac{3\pi}{4} = -\dfrac{3\sqrt{2}}{2}$,

$y = 4\sin\dfrac{3\pi}{4} = 2\sqrt{2}$, and $\dfrac{dy}{dx} = -\dfrac{4}{3}\cot\dfrac{3\pi}{4} = \dfrac{4}{3}$.

The equation of the tangent line is

$y = \dfrac{4}{3}\left(x + \dfrac{3\sqrt{2}}{2}\right) + 2\sqrt{2}$, or $y = \dfrac{4}{3}x + 4\sqrt{2}$.

51. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{5\sec^2 t}{3\sec t\tan t} = \dfrac{5\sec t}{3\tan t} = \dfrac{5}{3\sin t}$

At $t = \dfrac{\pi}{6}$, we have $x = 3\sec\dfrac{\pi}{6} = 2\sqrt{3}$,

$y = 5\tan\dfrac{\pi}{6} = \dfrac{5\sqrt{3}}{3}$, and $\dfrac{dy}{dx} = \dfrac{5}{3\sin\left(\frac{\pi}{6}\right)} = \dfrac{10}{3}$.

The equation of the tangent line is

$y = \dfrac{10}{3}(x - 2\sqrt{3}) + \dfrac{5\sqrt{3}}{3}$, or $y = \dfrac{10}{3}x - 5\sqrt{3}$.

52. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1+\cos t}{-\sin t}$

At $t = -\dfrac{\pi}{4}$, we have $x = \cos\left(-\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$,

$y = -\dfrac{\pi}{4} + \sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{\pi}{4} - \dfrac{\sqrt{2}}{2}$, and

$$\frac{dy}{dx} = \frac{1+\cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)} = \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2}+1.$$

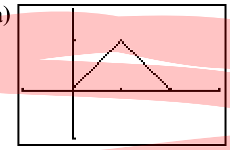
The equation of the tangent line is

$y = (\sqrt{2}+1)\left(x - \dfrac{\sqrt{2}}{2}\right) - \dfrac{\pi}{4} - \dfrac{\sqrt{2}}{2}$, or

$y = (1+\sqrt{2})x - \sqrt{2} - 1 - \dfrac{\pi}{4}$.

This is approximately $y = 2.414x - 3.200$.

53. (a)



[-1, 3] by [-1, 5/3]

(b) Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

(c) No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is -1.

54. (a) The function is continuous for all values of *m*, because the right-hand limit as $x \to 0$ is equal to $f(0) = 0$ for any value of *m*.

(b) The left-hand derivative at $x = 0$ is $2\cos(2\bullet 0) = 2$, and the right-hand derivative at $x = 0$ is *m*, so in order for the function to be differentiable at $x = 0$, *m* must be 2.

55. (a) For all $x \neq 0$ **(b)** At $x = 0$

(c) Nowhere

56. (a) For all x **(b)** Nowhere

(c) Nowhere

57. Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x-3) = -3$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-3) = -3$. Since these values agree with $f(0)$, the function is continuous at $x = 0$. On the other hand,

$f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is undefined at $x = 0$.

(a) $[-1, 0) \cup (0, 4]$ **(b)** At $x = 0$

(c) Nowhere in its domain

58. Note that the function is undefined at $x = 0$.

(a) $[-2, 0) \cup (0, 2]$ **(b)** Nowhere

(c) Nowhere in its domain

59.